COLLEGE ALGEBRA

W. L. HART

$x^3 \pm y^3 = (x^2 - xy + y^2)(x \pm y)$

$x^2 \pm 2xy + y^2 = (x \pm y)(x \pm y)$

$x^2 - y^2 = (x+y)(x-y)$

$ax \pm ay = a(x \pm y)$

$ax + ay + bx + by$

Prove $\div 0$ = impossible

division of a polynomial by a second order "

$0 = 0$

$a = 0 \quad b = 0$

$\frac{a \cdot 0}{0} = \frac{b \cdot 0}{0}$

$a = b$ one no. another no.

special products

$b(x+y) = bx + by$

$(x-y)(x+y) = x^2 + xy^2$

$(x+a)(x+b) = x^2 + (a+b)x + ab$

$(ax+b)(cx+d) = acx^2 + (bc+da)x + bd$

$(x+y)(x^2 + xy + y^2) = x^3 + y^3$

0 as exponent

$a^0 \cdot a^m = a^{m+0} = a^m$

$a^0 = \frac{a^m}{a^m} = 1$

$a^{-m} = \frac{1}{a^m}$

$a^{-2} = \frac{1}{a^2}$

$\left(\frac{a}{b}\right)^4 = (a^2)^2$

$(xyz)^0 = 1$

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~~Mt. Morris College~~
~~Mt. Morris~~

COLLEGE ALGEBRA

BY

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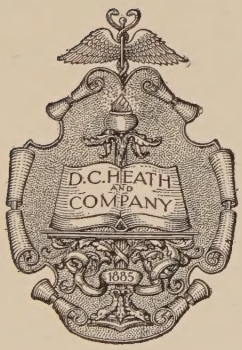

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PREFACE

THIS book is a flexible and distinctly modern text containing material for a rich course in algebra for college students who have had either two or three semesters of high school algebra. For a homogeneous group of students of any given preparation, there is an appropriate starting place in the book from which the group can proceed without digression through a unified course.

The early chapters give a complete treatment of elementary topics for those students who have had only two semesters of high school algebra. At the end of each of the 4th, 8th, and 12th chapters, there is a long list of review problems on the preceding four chapters, to provide material for students who need a review but not a complete treatment of the corresponding topics. In the later chapters, one finds all of the material ordinarily included in courses in college algebra and, besides, certain topics such as the mathematics of investment, infinite series and partial fractions, which are more rarely included.

Attention is directed to the following features:

Supplementary material. Difficult material which is not essential in a *minimum* course, is placed in sections and exercises, clearly marked *Supplementary*, whose omission in whole or in part does not disturb the continuity of the remainder of the text.

Arrangement of theoretical discussions. The proofs set a high standard of logical accuracy and each proof is organized into definite, easily verified steps. Each theorem is clearly illustrated, and its proof is displayed apart from the neighboring material, so that, when desired, an instructor can conveniently direct his students to read the illustrations and to omit the proof.

Adaptability for use where students are sectioned according to ability. The supplementary material, the arrangement of

theoretical discussions, and the careful grading of problems in all exercises, make it easy to elect either a minimum course giving merely the essentials or else a course of any desired richness.

Answers are provided only for odd-numbered problems in the exercises, so as to offer the instructor the greatest latitude. In each exercise the problems are so numerous and are arranged in such a manner that either the odd-numbered problems, or the even-numbered ones, by themselves, form a satisfactory exercise.

A well-organized graphical method for the solution of equations, in addition to Horner's method, is given in the chapter on the theory of equations. The text is arranged so that, if the instructor chooses to assign the graphical method, then Horner's method and its auxiliary theorems can be conveniently omitted.

The treatment of quadratic equations in one unknown is divided into two chapters to facilitate the omission of the elements of quadratics when they are not necessary in the course.

The chapter on the mathematics of investment gives a brief and yet complete treatment of the *simplest* annuity formulas and all of their usual applications. By restricting the discussion to this simplest case, where problems can be solved by use of the appended interest tables, the author is able to exhibit the innate simplicity and power of the subject without bewildering the student with artificial and unusual logarithmic computations.

The chapter on logarithms gives the instructor the option of requiring computation with either five-place or four-place logarithms.

In the chapter on infinite series, the definition of convergence and extensive numerical applications are given at the beginning, so that they can be considered even though the students are to omit tests for convergence and elementary facts about limits.

Determinants of the second and third orders are given a separate treatment apart from the general theory of determinants.

The advanced chapters have been made particularly complete and intelligible, and they possess the mutual independence desirable in this part of a college algebra.

Numerical applications and checks are emphasized at every opportunity.

Illustrative examples are used extensively to illustrate new theory, to recall previous knowledge, and to furnish models for the students' own solutions.

Miscellaneous problems are given at the end of almost every chapter.

Frequent and clearly displayed summaries are introduced to aid the student in future references and to systematize his methods.

The author acknowledges his indebtedness to Professors William H. Bussey, Dunham Jackson and Anthony L. Underhill of the University of Minnesota, and to Prof. Walter W. Hart of the University of Wisconsin, for many valuable suggestions.

University of Minnesota
December 1, 1925

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COLLEGE ALGEBRA

CHAPTER I

ELEMENTARY TOPICS

1. The fundamental operations of algebra are addition, subtraction, multiplication, and division. The elementary properties of these operations are consequences of the following properties of addition and multiplication, which are assumed as a foundation for all algebraic operations.

A. The commutative law for addition. — *The sum of two or more numbers is the same, in whatever order they are added. Thus,* $a + b + c = a + c + b$.

B. The associative law for addition. — *The sum of three or more numbers is the same in whatever manner the numbers are grouped. Thus,* $a + b + c = a + (b + c) = (a + b) + c$.

C. The commutative law for multiplication. — *The product of two or more numbers is the same in whatever order the numbers are arranged. Thus,* $ab = ba$.

D. The associative law for multiplication. — *The product of three or more numbers is the same in whatever manner the numbers are grouped. Thus,* $abc = a(bc)$.

2. Operations involving the number zero. — If N is any number, then $N \cdot 0 = 0$.

Thus, $7 \cdot 0 = 0$; $(-3) \cdot 0 = 0$; $a \cdot 0 = 0$.

Division by zero is not allowed in algebra because contradictory results would be obtained if division by zero were permitted.

Thus, if we were to define $\frac{5}{0}$ as the number X which, when multiplied by 0, gives 5, it would follow that $5 = 0 \cdot X$. Hence, since $0 \cdot X = 0$, we should have $5 = 0$, which is a contradiction.

3. The laws of signs. — When no sign is indicated at the left of a number, the sign is understood to be $+$. Thus, "5" means $+5$.

By definition, the **absolute value** of a positive number is the number itself. The absolute value of a negative number is the given number with its sign changed to $+$.

Thus, the absolute value of 5 is 5; of -18 is 18; of -5 is 5.

To add two numbers having the same sign, *add their absolute values and attach their common sign.*

Thus, $(-5)+(-7)=-12$; $(+6)+(+13)=+19$.

To add two numbers having unlike signs, *subtract the smaller absolute value from the larger and prefix to the result the sign of the number having the larger absolute value.*

Thus, $(+2)+(-9)=-7$; $(+16)+(-5)=+11$.

To subtract one number from another, *change the sign of the subtrahend and add the result to the minuend.*

Thus, $(-6)-(+4)=-6+(-4)=-10$;
$(-5)-(-4)=(-5)+(+4)=-1$.

The sign of the product or the quotient of two numbers:

1. *Give the result a plus sign if the two numbers have like signs.*
2. *Give the result a minus sign if the two numbers have unlike signs.*

Thus, $(-6)\cdot(-7)=+42$; $(-5)\cdot(+7)=-35$; $\frac{-40}{+4}=-10$.

NOTE. — In this book, the **difference** of two numbers means *the first number minus the second.*

EXERCISE 1

Find the sum, the difference, the product, and the quotient of the first number divided by the second, for each of the following pairs of numbers:

1. 3 and 2.	**4.** 39 and -3.	**7.** 6 and -18.
2. -15 and 3.	**5.** -16 and -4.	**8.** 0 and 12.
3. 12 and -4.	**6.** 32 and -4.	**9.** -5 and -2.

10. What is the sign of the product of an even number of negative factors; of an odd number of negative factors?

4. Parentheses ( ), **brackets**, [ ], and **braces**, { }, are symbols of grouping used to indicate terms which are to be treated as a single number expression.

To remove parentheses preceded by a plus sign, *rewrite all of the included terms without changing their signs.*

To remove parentheses preceded by a minus sign, *rewrite all of the included terms, changing their signs from* $+$ *to* $-$, *or from* $-$ *to* $+$.

Thus, $x-(3y-4z+5)+(4a-b)=x-3y+4z-5+4a-b$.

To inclose terms within parentheses preceded by a plus sign, *rewrite the terms without changing their signs.*

To inclose terms within parentheses preceded by a minus sign, *rewrite the terms, changing their signs from* $+$ *to* $-$, *or from* $-$ *to* $+$.

Thus, $-3+5b-x=-(3-5b+x)$.

If several symbols of grouping are involved together, remove them by removing **the innermost one first**, and so on until the last one is removed.

Thus, $b-\{3y-[(x+2)-4z]\}=b-\{3y-[x+2-4z]\}$
$=b-\{3y-x-2+4z\}=b-3y+x+2-4z$.

EXERCISE 2

Remove all signs of grouping and combine similar terms:

1. $6\,a - [3\,a - (4\,a - 1)]$.

2. $(7\,t + 3) - [5\,t + (2\,t - 3)]$.

3. $2.25\,x - [4\,a - (6.3\,x - .4)]$.

4. $(8\,m - .3\,m + 2) - (3\,m - .5)$.

5. $7\,z - [-\,z - (z + 1)]$.

6. $7\,c - [(3\,d + 8\,c) - (d - c)]$.

7. $(h - x) - [(5 + 2\,x) - 3\,x]$.

8. $-[5 - (x + 2)] - \{[x - (x - 2)] + 1\}$.

9. $2.4\,y + [7.2\,y - (3\,y - 2.4)]$.

10. $4\,t - [(3\,t - 5) + 6\,t - 5]$.

11. $(x - 2\,y) - [5 - (x + 3\,y - 1)]$.

12. $(x - 4) - [-\,1 + (3 - x)]$.

13. Find the sum and the difference of $(4\,x - 3\,y + 5\,z)$ and $(2\,y - 3\,x - 4\,z)$.

HINT. — Their sum is $(4\,x - 3\,y + 5\,z) + (2\,y - 3\,x - 4\,z)$; their difference is $(4\,x - 3\,y + 5\,z) - (2\,y - 3\,x - 4\,z)$. In each case remove parentheses and combine like terms.

14. Find the sum and the difference of $(3\,b^2 - 4\,ab)$ and $(2\,b^2 + 8\,ab - a^2)$.

15. What must be added to 10 to give 17; to $(3\,x^2 + 5\,x - 7)$ to give $(5\,x^2 - 2\,x + 8)$?

16. What must be added to $(5\,y^2 - 6\,xy + 8\,x^2)$ to give $(3\,y^2 - 8\,xy)$?

17. What must be subtracted from 18 to give 12; from $(4\,x^3 - 2\,x^2 + 5\,x - 2)$ to give $(7\,x^3 - 5\,x^2 - x - 7)$?

18. Add $(5\,x^2 - xy + 7\,y^2)$ to the difference of $(xy - y^2)$ and $(3\,y^2 - x^2)$.

5. Exponents. — We write a^2 to abbreviate $a \cdot a$; a^3 to abbreviate $a \cdot a \cdot a$. In general, if m is any positive integer, we define a^m by the equation,

$$a^m = a \cdot a \cdot a \cdots a. \quad (m \text{ factors}) \tag{1}$$

In equation 1, we call a the **base** and m the **exponent.** *An exponent tells how many times the base is used as a factor.*

Thus, $(-5)^3 = (-5)(-5)(-5) = -125$.

Until otherwise specified, a literal number, used as an exponent, represents a positive integer.

6. Law of exponents for multiplication. — *The exponent of the product of two powers of the same base is the sum of the exponents of the two powers.* That is,

$$a^m a^n = a^{m+n} \tag{1}$$

Proof. — **1.** By definition, $a^m = a \cdot a \cdot a \cdots a.$ $(m \text{ factors})$
$a^n = a \cdot a \cdot a \cdots a.$ $(n \text{ factors})$

2. Therefore, $a^m a^n = (\underbrace{a \cdot a \cdot a \cdots a}_{m \text{ factors}})(\underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}})$.

3. $a^m a^n = a \cdot a \cdot a \cdots a \cdot a \cdot a \cdot a \cdots a.$ $(m + n \text{ factors})$

4. Hence, by the definition of an exponent, $a^m a^n = a^{m+n}$.

Illustration. $a^7 a^3 = a^{10}$; $(25)^4(25)^2 = (25)^6$.

7. Multiplication of polynomials. — An algebraic expression containing only one term is called a **monomial**, and one containing two or more terms is called a **polynomial.** A polynomial with two terms is called a **binomial**, and one with three terms is called a **trinomial.**

To multiply two monomials, proceed as follows:

1. *Find the product of their numerical coefficients.*

2. *Multiply this result by the product of the literal parts, using the law of exponents for multiplication.*

Illustration. $(16\,xy^3)(-2\,x^2y) = -32(xy^3)(x^2y) = -32\,x^3y^4$.

To multiply a polynomial by a monomial, *multiply each term of the polynomial by the monomial and combine the results.*

Illustration. $5x(3x^3 - 4y - 5z) = 15x^4 - 20xy - 25xz.$

To multiply a polynomial by a polynomial:

1. *Multiply the multiplicand by each term of the multiplier.*
2. *Add the results of Step* 1.

When possible, arrange both multiplier and multiplicand in descending, or ascending, powers of one letter.

Example 1. — Multiply $(x^2 - y^2 + 2xy)$ by $(y^2 + x^2 - 2xy)$.

Solution. —

$$\begin{array}{l} x^2 + 2xy - y^2 \quad . \quad . \quad . \quad . \quad . \quad = 1 + 6 - 9 = -2. \\ \underline{x^2 - 2xy + y^2 \quad . \quad . \quad . \quad . \quad . \quad = 1 - 6 + 9 = 4.} \\ x^4 + 2x^3y - x^2y^2 \\ \quad - 2x^3y - 4x^2y^2 + 2xy^3 \\ \qquad\qquad x^2y^2 + 2xy^3 - y^4 \\ \hline x^4 \qquad - 4x^2y^2 + 4xy^3 - y^4 \quad \cdots = 1 - 36 + 108 - 81 = -8. \end{array}$$

Check. — Place $x = 1$ and $y = 3$:

$$\left.\begin{array}{l} \\ \end{array}\right\} (-2)(4) = -8.$$

When convenient, multiplication of polynomials should be performed as in the following example.

Example 2. — Find $(2x - 3y)(x^2 - 3xy - 5y^2)$.

Solution. — 1. $(2x - 3y)(x^2 - 3xy - 5y^2)$

2. $= 2x(x^2 - 3xy - 5y^2) - 3y(x^2 - 3xy - 5y^2)$

3. $= 2x^3 - 6x^2y - 10xy^2 - 3x^2y + 9xy^2 + 15y^3$

4. $= 2x^3 - 9x^2y - xy^2 + 15y^3.$

EXERCISE 3

Multiply:

1. $(4x^3)(5x^2)$.

2. $(8y^2)(3xy^3)$.

3. $x(5 - 3y)$.

4. $3(x - 2y)$.

5. $(2x)(3x^3y)$.

6. $(4xy)(8x^2y^3)$.

7. $x^2(4 - 5xy)$.

8. $x^3(-2 - 3x)$.

9. $(-3xy^2)(5xyz)$.

10. $(-2a^3b^2)(-3ab^4)$.

11. $-2x(5x - 7y + z)$.

12. $-5x^3(x^2 - y^2 - 2xy)$.

13. $2x(-x^2 - 5 - 3xy)$.

Multiply and check as in illustrative Example 1, *of Section* 7:

14. $(2 - 5x)(7 - x - x^2)$. **15.** $(x - 3)(x^2 + 5 - 7x)$.

16. $(x^3 - x^2 - x - 1)(x^2 - 2)$.

17. $(p^2 - pr + r^2)(pr + r^2 + p^2)$.

18. $(x^2 - 3x + 4)(2x^2 - x + 3)$.

19. $(5a^2 - 6ab - b^2)(4ab - b^2 - a^2)$.

20. $(2z - 5wz - z^2)(6z^2 - 3w^2 + wz)$.

Multiply as in illustrative Example 2, *of Section* 7:

21. $(2x^2 - 5x + 7)(x - 3)$.

22. $(2x^2 + xy - 4y^2)(x - 2y)$.

23. $(5x^2 - 5xy - 3y^2)(-x - 5y)$.

24. $(3 - 5x)(4 - x - x^3)$. **25.** $(x + a)(x + b)$.

26. $(a^2 - ab - 5b^2)(5a - 2b)$.

27. $(9a^2 - 3ab + b^2)(3a + b)$.

28. $(4x^2 - 5y^2)(4x^2 + 5y^2)$.

29. $(a - b)(a^2 - ab + b^2)$. **30.** $(cx + d)(ax + b)$.

31. Multiply $(7x - 3y - 5)$ by $(4y - 2x - 3)$.

32. Multiply $(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$ by $(x - y)$.

Perform indicated operations and remove all signs of grouping:

33. $(x - 2y)(x + y) - 3x(x - y) - 5y(x - y)$.

34. $(x^2 + y^2 - xy)(x + y) - x^2(x + y^2) - y^2(x^2 - y)$.

35. $(3a - b)(3a + b) - 3(a - b) - (a - b)(a + b)$.

8. Law of exponents for division. — If m is greater than n,

$$\frac{a^m}{a^n} = a^{m-n}. \tag{1}$$

That is, *the exponent in the quotient of two powers of the same base is the exponent of the dividend minus the exponent of the divisor.*

Proof. — **1.** $\dfrac{a^m}{a^n} = \dfrac{a \cdot a \cdot a \cdots a \cdot a \cdot a \cdot a \cdots a}{a \cdot a \cdot a \cdots a}$ $\begin{matrix}(m \text{ factors}) \\ (n \text{ factors})\end{matrix}$

$= a \cdot a \cdots a.$ $[(m - n) \text{ factors}].$

2. Therefore, by the definition of an exponent, $\dfrac{a^m}{a^n} = a^{m-n}$.

Illustration. $\dfrac{a^{17}}{a^4} = a^{13}$, $\dfrac{2^6}{2^4} = 2^2 = 4$.

9. Division by polynomials.

To divide a monomial by a monomial:

1. *Find the quotient of the numerical coefficients.*

2. *Multiply this result by the quotient of the literal parts of the monomials, using the law of exponents for division.*

Illustration. $\dfrac{16\,a^3b^2}{-4\,ab} = \dfrac{16}{-4} \cdot \dfrac{a^3b^2}{ab} = -4\,a^2b.$

To divide a polynomial by a monomial, *divide each term of the polynomial by the monomial and unite the results with proper signs.*

Thus, $\dfrac{-3x^4 + 6x^2 - x^3}{-x^2} = \dfrac{-3x^4}{-x^2} + \dfrac{6x^2}{-x^2} + \dfrac{-x^3}{-x^2} = 3x^2 - 6 + x.$

To divide one polynomial by another:

1. *Arrange both dividend and divisor in either ascending or descending powers of some common letter.*

2. *Divide the first term of the dividend by the first term of the divisor and write the result as the first term of the quotient.*

3. *Multiply the whole divisor by the first term of the quotient and subtract the product from the dividend.*

4. *Consider the remainder obtained in Step* 3 *as a new dividend and repeat Steps* 1 *to* 3; *etc.*

NOTE 1. — The following relation is true in division:

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}. \qquad (1)$$

Or, $$\text{dividend} = (\text{quotient})\,(\text{divisor}) + \text{remainder}. \qquad (2)$$

Example 1. — Divide

$$(2\,a^4 + 8\,a - a^3 + 15) \text{ by } (2\,a^2 - 3\,a + 5).$$

Solution. —

$$\begin{array}{r|l} & a^2 + a - 1 \\ \hline 2\,a^2 - 3\,a + 5 & 2\,a^4 - a^3 + 8\,a + 15 \\ & \underline{2\,a^4 - 3\,a^3 + 5\,a^2} \\ & \quad 2\,a^3 - 5\,a^2 + 8\,a + 15 \\ & \quad \underline{2\,a^3 - 3\,a^2 + 5\,a} \\ & \quad\quad - 2\,a^2 + 3\,a + 15 \\ & \quad\quad \underline{- 2\,a^2 + 3\,a - 5} \\ & \quad\quad\quad + 20 \end{array}$$

Check. — When $a = 2$: dividend = 55; divisor = 7; quotient = 5; remainder = 20. We test these values by equation 2 of Note 1: Does $55 = 7 \cdot 5 + 20$? Yes.

Note. — A numerical check as in Example 1 will detect practically any error in division. To obtain an absolute check, we should multiply the divisor by the quotient, add the remainder, and notice whether we thus obtain the dividend.

EXERCISE 4

Divide :

1. $-10\,ab^2$ by $2\,ab$.

2. $24\,x^3y^2z$ by $-6\,xy^2z$.

3. $-9\,xyb^2$ by $3\,xyb$.

4. $(3\,x^2 - x + 7\,x^3)$ by x.

5. $(x^2b - xb - b^2)$ by $-b$.

6. $(36\,abc^2 - 24\,a^2bc)$ by $-12\,abc$.

7. $(32\,ab^4 - 48\,ay^4)$ by $16\,a$.

8. $(my^2 - 6\,my + 9\,y)$ by $-y$.

9. $(21\,a^3b^2 - 42\,ab^3)$ by $-7\,ab^2$.

10. $(-3\,x^4 + 6\,x^2 - x^3)$ by $-x^2$.

Perform indicated divisions and check by substitution :

11. $(a^2 + 5\,a + 6) \div (a + 3)$.

12. $(x^2 + 8\,x + 12) \div (x + 2)$.

13. $(m^2 + 10\,m + 10) \div (m + 1)$.

14. $(x^2 - 12\,x + 39) \div (x + 7)$.

15. $(n^4 - 5\,n^2 - 24) \div (n^2 + 3)$.

16. $(x^2 - 2xz - 48z^2) \div (x - 8z)$.

17. $(x^4 + 6x^2y - 28y^2) \div (x^2 - 3y)$.

18. $(x^2y^2 - 18xy + 45) \div (xy - 3)$.

19. $(x^4 - x^3y - 2x^2y^2 - 3xy^3 - y^4) \div (x^2 + xy + y^2)$.

20. $(15x^3 - 30x - 8 - 19x^2) \div (-5x + 3x^2 - 4)$.

21. $\dfrac{-2m^2 + 4mn - 6n^2}{-2}$.

22. $\dfrac{16a^3b - 12a^2b^2 + 8ab^3}{-2b}$.

23. $\dfrac{5n^3 + 8n^2 - 1 - 23n}{5n^2 - 7n - 2}$.

24. $\dfrac{2a^3 - 9a^2b + 17ab^2 - 12b^3}{2a - 3b}$.

25. $\dfrac{8a^3 - 27}{2a - 3}$.

26. $\dfrac{6x^4 - 7x^3 - 28x^2 + 8x - 21}{3x^2 - 5x - 7}$.

27. $\dfrac{x^5 + y^5}{x + y}$.

28. $\dfrac{27x^3 + 125y^3}{3x - 5y}$.

29. $\dfrac{z^3 - 8}{z + 2}$.

30. $\dfrac{12a^3 + 17a^2b - 25b^3 - 20ab^2}{3a + 5b}$.

CHAPTER II

SPECIAL PRODUCTS AND FACTORING

10. Special products. — The reader should verify the following type products by direct multiplication:

Type I: $a(x + y) = ax + ay.$

Type II: $(x + y)(x - y) = x^2 - y^2.$

Type III: $(a + b)^2 = a^2 + 2ab + b^2.$

Type IV: $(a - b)^2 = a^2 - 2ab + b^2.$

Type V: $(x + a)(x + b) = x^2 + (a + b)x + ab.$

Type VI: $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd.$

These type products should be used as formulas.

$$(3b - 2b^2)(3b + 2b^2) = (3b)^2 - (2b^2)^2 = 9b^2 - 4b^4. \quad \text{(Type II)}$$

$$\begin{aligned}(5c - 2k)^2 &= (5c)^2 - 2(5c)(2k) + (2k)^2 \\ &= 25c^2 - 20ck + 4k^2. \quad \text{(Type IV)}\end{aligned}$$

$$\begin{aligned}(4x + 5)(-2x + 3) &= -8x^2 + (12 - 10)x + 15 \quad \text{(Type VI)} \\ &= -8x^2 + 2x + 15.\end{aligned}$$

NOTE. — Remember types II, III, and IV in words. Thus, type IV states that *the square of the difference of two numbers equals the square of the first number, minus twice the product of the numbers, plus the square of the second number.* The *sum* $(a + b)x$, in type V, and $(ad + bc)x$ in type VI, may be remembered as *the sum of the cross products.*

EXERCISE 5

Find the following products and powers mentally:

1. $6c(4 - 5a).$
2. $(w + s)^2.$
3. $(u - v)^2.$
4. $-5x(a - h).$
5. $(2x + 2)(x - 3).$
6. $(2x - 3y)(2x + 3y).$
7. $(4x + 5y)^2.$
8. $(3a - b)^2.$
9. $(7a - b)(6a + b).$
10. $(3x + .2)(x - 1).$
11. $(x - \frac{3}{5})(x + \frac{3}{5}).$
12. $(b^2 - 3ac)(2b^2 + 5ac).$

13. $(1 - 5a^2)(2 + 7a^2)$.
14. $(12 - 1)(12 + 1)$.
15. $(100 - 2)(100 - 3)$.
16. $(95)(105)$.
17. $(x^2 - 2y^2)(x^2 + 2y^2)$.
18. $(\frac{1}{3}s - 4t)(\frac{1}{3}s + 4t)$.
19. $(8m - 5n)(3m + 4n)$.
20. $(3x - \frac{1}{2})^2$.
21. $(-2 - 5w^2)(-2 + 5w^2)$.
22. $(4p - 7r^2)(3p + 5r^2)$.
23. $(x - 8c)(x + 10c)$.
24. $(5yz - 2)(9yz - 4)$.
25. $(3x + 5y)(2x - 7y)$.
26. $(x + \frac{1}{2})^2$.
27. $(2x - \frac{1}{4})^2$.
28. $(3x + 2)(-x - 5)$.
29. $(11c^2 - 8d)(11c^2 + 8d)$.
30. $(\frac{1}{5}x - 2y)(\frac{1}{5}x + 2y)$.
31. $(x^2y - 3a)(x^2y + 5a)$.
32. $(-9t^2 + a)(8t^2 - 2a)$.
33. $(-x - y)^2$.
34. $(x - 4a)^2$.
35. $(x^2 - 4y^3)(x^2 + 4y^3)$.
36. $(4x + y)(3x - y)$.
37. $(7z + x^2y)(3z + 2x^2y)$.
38. $(2y + 10z)(y - 4z)$.
39. $(2x - 3y)(2x + 4y)$.
40. $(5w^3 - 3x^2)(5w^3 - 3x^2)$.

41. Multiply $(c + 2d - 11a)(c + 2d + 11a)$.

Solution. — 1. $(c + 2d - 11a)(c + 2d + 11a)$
$= [(c + 2d) - 11a][(c + 2d) + 11a)]$
2. (Type II) $= (c + 2d)^2 - (11a)^2 = c^2 + 4cd + 4d^2 - 121a^2$.

Find the following products by use of types II to VI:

42. $[(a + k) + 5][(a + k) - 5]$.
43. $[(h + w) - z][(h + w) + z]$.
44. $[3p + (c + d)]^2$.
45. $[h - (k - w)]^2$.
46. $[(x + y) + 2][(x + y) + 7]$.
47. $[(a + b) - 5][(a + b) + 7]$.
48. $(a - b + d)(a - b - d)$.
49. $(x - y - z)(x - y + z)$.
50. $(x + 2y - 3z)(x + 2y + 3z)$.
51. $[(a + b) + (c + d)][(a + b) - (c + d)]$.
52. $(m + n - 4)(m + n - 8)$.
53. $(x - y + 8)(x - y - 6)$.
54. $(3x - y + z)(2x - y + z)$.
55. $(x + 3y + 15z)(x + 3y - 10z)$.
56. $(r + 2s - 3t)(r + 2s + 7t)$.
57. $(x^2 + 2x + 1)(x^2 + 2x - 5)$.
58. $[(x + y) + 4]^2$.
59. $(x + y + z)^2$.

60. From the result of Problem 59, state in words a rule for finding the square of the sum of any three numbers.

By use of the result of Problem 60, *expand each expression :*

61. $(a + b + y)^2$. **62.** $(k + 3x + 4w)^2$. **63.** $(3z + 5 - x)^2$.

11. Integral rational expressions. — A term is **integral** and **rational** in certain letters if the term is a product of positive integral powers of the letters, multiplied by some coefficient. An **integral rational polynomial** in certain letters is one in which all terms are integral and rational.

Thus, $(3x^2 - \frac{1}{2}yz^2)$ is an integral rational polynomial in x, y, and z; $\left(5z\sqrt{x} + \frac{1}{y}\right)$ is integral and rational in z, but not in x or y.

Unless otherwise specified, **to factor** an integral rational polynomial will mean to find two or more *integral rational polynomials* whose product is the given polynomial.

12. Factoring by inspection.

Type A. **A polynomial with a monomial factor :**

$$kx + ky + kz = k(x + y + z).$$

Type B. **The difference of two squares :**

$$x^2 - y^2 = (x - y)(x + y).$$

Illustration. $25x^2 - \frac{y^2}{4} = \left(5x - \frac{y}{2}\right)\left(5x + \frac{y}{2}\right)$.

Type C. **A perfect square trinomial :**

$a^2 + 2ab + b^2 = (a + b)^2$, and $a^2 - 2ab + b^2 = (a - b)^2$.

Illustration. $4x^2 - 20xy + 25y^2 = (2x - 5y)^2$.

NOTE. — **In a perfect square trinomial,**

1. *two terms are perfect squares, and*

2. *the third term is plus, or minus, twice the product of the square roots of the other terms.*

Type D. Certain trinomials of the form $gx^2 + hx + k$.

Example 1. — Factor: $x^2 - 2x - 8$.

Solution. — 1. We wish to find numbers a and b so that

$$(x + a)(x + b) = x^2 + (a + b)x + ab = x^2 - 2x - 8.$$

2. Hence, we seek numbers a and b such that $a + b = -2$ and $ab = -8$. Since $ab = -8$, a and b have opposite signs.

3. Try $a = -2$ and $b = 4$; since $(-2 + 4) = 2$, this choice is incorrect.

4. Try $a = 2$ and $b = -4$; $x^2 - 2x - 8 = (x + 2)(x - 4)$.

Example 2. — Factor: $15x^2 + 2x - 8$.

Solution. — 1. We wish to find a, b, c, and d so that

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd = 15x^2 + 2x - 8.$$

2. Hence, $ac = 15$, $bd = -8$, and $(ad + bc) = 2$.

3. Since $ac = 15$, try $a = 15$ and $c = 1$; since $bd = -8$, try $b = 2$ and $d = -4$. This choice is incorrect because

$$(15x + 2)(x - 4) = 15x^2 - 58x - 8.$$

4. Try $a = 3$, $c = 5$, $b = -2$, and $d = 4$. This choice is correct because $(3x - 2)(5x + 4) = 15x^2 + 2x - 8$.

13. Prime factors. — An integral rational polynomial is **prime** if it has no integral rational factors except itself and 1.

Thus, $(x - y)$ is prime although $(x - y) = (\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$, because these two factors are not integral and rational.

In the future, unless otherwise stated, **to factor a polynomial will mean to find all of its prime factors.**

NOTE. — Factoring should always be checked by mental multiplication of the factors found.

EXERCISE 6

Factor by use of formula B, Section 12:

1. $y^2 - 49x^2$.
2. $9x^2 - 1$.
3. $49x^2 - 625z^2$.
4. $36r^4 - 121s^2$.
5. $4a^4 - 25b^2$.
6. $1 - 100b^6z^4$.
7. $w^8 - z^8$.
8. $-81 + 4x^2$.
9. $64y^2 - 36a^2b^2$

Factor by use of formula C, Section 12:

10. $x^2 - 4x + 4$.
11. $16 + 8y + y^2$.
12. $4x^2 - 12xy + 9y^2$.
13. $w^2 + 10w + 25$.
14. $a^2x^2 - 18ax + 81$.
15. $a^2 - a + \frac{1}{4}$.
16. $x^2 + \frac{2}{5}x + \frac{1}{25}$.
17. $m^2 + 8mn + 16n^2$.
18. $36z^2 - 24zw + 4w^2$.
19. $1 + 4xy + 4x^2y^2$.

Factor, as in illustrative Examples 1 *and* 2, *Section* 12:

20. $x^2 + 14x + 33$.
21. $t^2 - 10t + 21$.
22. $x^2 - 10x + 24$.
23. $w^2 + 4w - 60$.
24. $6a^2 - a - 1$.
25. $2x^2 - 15x + 27$.
26. $8a^2 - 22ab + 15b^2$.
27. $7r^2 + 4rs - 11s^2$.
28. $12t^2 - 13t + 3$.
29. $2 - 7y + 6y^2$.

Find the prime factors of each expression:

30. $3cz^2 - 22cz + 7c$.
31. $x^4 + 27x^2 + 50$.
32. $12a^2b - 16ab^2$.
33. $25c^2 - 10cd + d^2$.
34. $t^2 - st - 20s^2$.
35. $7ab - 60b^2 + a^2$.
36. $3 - 12t^2 - 9t$.
37. $16x^4 - 81y^4$.
38. $a^2b + 8ab - 20b$.
39. $2rw^2 - 3rw - 20r$.
40. $9 + 6(y + x) + (y + x)^2$.
41. $(a + x)^2 - 4(a + x) + 4$.
42. $(x - y)^2 - 16(y + x)^2$.
43. $\frac{1}{4} - 36a^2b^2$.
44. $c^4x - 16d^4x$.
45. $9m^2n^2 + 25r^4 - 30mnr^2$.
46. $x^2y^2 - 16axy + 28a^2$.
47. $52 - a^2b^2 - 9ab$.
48. $10n^2 + 19nt + 9t^2$.
49. $132c - 135 + 3c^2$.

50. $(r + s)^2 + 4(r + s)t - 5t^2$.

Solution. — Just as $y^2 + 4yz - 5z^2 = (y - z)(y + 5z)$, so
$(r + s)^2 + 4(r + s)t - 5t^2 = [(r + s) - t][(r + s) + 5t]$.

Find the prime factors of each expression:

51. $(a + b)^2 + 8(a + b) - 20$.
52. $(x + y)^2 + 23(x + y) + 60$.
53. $c^2 - (a + b)^2$.
54. $(a - 2b)^2 - 3z(a - 2b) - 88z^2$.

55. $96\,z^2 - 20(a + b)z - (a + b)^2$.

56. $1 + 2(a^2 - 2\,a) + (a^2 - 2\,a)^2$.

57. $(m - n)^2 - (x + y)^2$.

58. $9(m - n)^2 - 12(m - n) + 4$.

59. $36 + 5(x + y) - (x + y)^2$.

60. $(7\,x - 2\,y)^2 - y^2$.

61. $16\,x^4 - 625\,y^4$.

62. $xz + 6\,z - x^2z$.

63. $(y - 4)^2 - (x - 2)^2$.

64. $m^2 - (n - p)^2$.

65. $(1 + n)^2 - 4\,m(1 + n) + 4\,m^2$.

66. $(x - y)^2 - 2(x - y) - 63$.

67. $(x^2 + x - 9)^2 - 9$.

68. $5 + 4(x + z) - (x + z)^2$.

69. $144 - 7(m + n) - (m + n)^2$.

70. $(a - c)^2 - 6\,b(a - c) + 9\,b^2$.

14. Factoring by grouping.

Example 1. — Factor: $6 - 3\,x^2 - 8\,x + 4\,x^3$.

Solution. — 1. $6 - 3\,x^2 - 8\,x + 4\,x^3 = (6 - 8\,x) - (3\,x^2 - 4\,x^3)$

2. $= 2(3 - 4\,x) - x^2(3 - 4\,x) = (2 - x^2)(3 - 4\,x)$.

Example 2. — Factor: $a^2 - c^2 + b^2 - d^2 - 2\,ab - 2\,cd$.

Solution. — 1. $a^2 - c^2 + b^2 - d^2 - 2\,ab - 2\,cd$

2. $= (a^2 - 2\,ab + b^2) - (c^2 + 2\,cd + d^2) = (a - b)^2 - (c + d)^2$

3. $= [(a - b) + (c + d)][(a - b) - (c + d)]$

4. $= (a - b + c + d)(a - b - c - d)$.

EXERCISE 7

Factor by grouping or by reducing to the difference of two squares:

1. $4\,h(x + y) - k(x + y)$.

2. $2\,m(x + y) - 3(x + y)$.

3. $5\,c(r - s) - 2\,d(r - s)$.

4. $9(p - q) - x(p - q)$.

5. $ay + az + 5\,y + 5\,z$.

6. $mr - ms + nr - ns$.

7. $ax - ay - bx + by$.

8. $5\,mr - 8\,ns - 8\,ms + 5\,nr$.

9. $3\,x^3 - 6\,x^2 + x - 2$.

10. $x^3 + x^2 + x + 1$.

11. $(a + b)^2 - c^2$.

12. $(m^2 - 2\,mn + n^2) - 4\,p^2$.

13. $y^2 - (x^2 + 6\,x + 9)$.

14. $t^2 - r^2 - 2\,rs - s^2$.

15. $4\,c^2 + 4\,c + 1 - b^2$.

16. $x^3 + x^2y - xy^2 - y^3$.

17. $m^3 - m^2n - mn^2 + n^3$.
18. $81z^2 + 20w - 4w^2 - 25$.
19. $4y^2 - z^2 + 4xy + x^2$.
20. $x^2 + y^2 - 2xy - k^2$.
21. $3y^3 + y^2 - 2 - 6y$.
22. $ac - 3bc - ad + 3bd$.
23. $2cd - c^2d - 6 + 3c$.
24. $9a^2 + 4bc - 4c^2 - b^2$.
25. $x^2 + 4x + 4 - y^2$.
26. $2ac - 4ay - bc + 2by$.
27. $2s^3 - 4s^2t + st^2 - 2t^3$.
28. $a^3 - a^2b + ab^2 - b^3$.
29. $40x^3 - 42 - 35x^2 + 48x$.
30. $4 - 22xy - 121x^2 - y^2$.
31. $r^2 - 9s^2 + 25 + 6st - 10r - t^2$.
32. $4a^2 - x^2 + 2xy - 12ab - y^2 + 9b^2$.
33. $6ab + a^2 - x^2 + 9b^2 - 4x - 4$.
34. $4a^2 + 1 - b^2 - 4a - 9 - 6b$.
35. $2cd + 25a^2 - c^2 - 10ab - d^2 + b^2$.

15. Trinomials reducible to the difference of two squares. Trinomials of the form $(x^4 + kx^2y^2 + y^4)$ can be reduced to the difference of two squares if k has such a value that $(x^4 + kx^2y^2 + y^4)$ becomes a perfect square after the addition of a positive multiple of x^2y^2.

Example 1. — Factor: $64a^4 - 64a^2b^2 + 25b^4$.

Solution. — 1. Recall that $(8a^2 + 5b^2)^2 = 64a^4 + 80a^2b^2 + 25b^4$.

2. Hence, $(64a^4 - 64a^2b^2 + 25b^4)$ becomes a perfect square if $144a^2b^2$ is added.

3. $(64a^4 - 64a^2b^2 + 25b^4)$
$= (64a^4 - 64a^2b^2 + 25b^4 + 144a^2b^2) - 144a^2b^2$
$= (64a^4 + 80a^2b^2 + 25b^4) - 144a^2b^2$
$= (8a^2 + 5b^2)^2 - 144a^2b^2$
$= (8a^2 + 5b^2 - 12ab)(8a^2 + 5b^2 + 12ab)$.

EXERCISE 8

Factor by reducing to the difference of two squares:

1. $a^4 + a^2b^2 + b^4$.
2. $m^4 + m^2 + 1$.
3. $x^4 + 4x^2 + 16$.
4. $25x^4 + 9x^2y^2 + y^4$.
5. $4a^4 + 8a^2 + 9$.
6. $r^4 + 64t^4$.
7. $16a^4 - 12a^2 + 1$.
8. $9a^4 + 2a^2b^2 + b^4$.

9. $9\,r^4 - 43\,r^2t^2 + 49\,t^4$.
10. $r^4 + 3\,r^2s^2 + 36\,s^4$.
11. $25\,z^4 - 19\,z^2w^2 + w^4$.
12. $9\,c^4 - 34\,c^2d^2 + 25\,d^4$.
13. $x^4 + 4$.
14. $36\,a^4 - 16\,a^2 + 1$.
15. $16\,x^4 + 4\,x^2y^2 + 25\,y^4$.
16. $16\,a^4 - 28\,a^2b^2 + 9\,b^4$.
17. $9\,x^4 - 10\,x^2 + 1$.
18. $4\,m^4 + 19\,m^2n^2 + 49\,n^4$.
19. $4\,a^4 + 1$.
20. $x^4 + 324$.

16. The sum and the difference of two cubes. — By multiplication we verify that

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2); \qquad (1)$$
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2). \qquad (2)$$

Notice that $(a^2 + ab + b^2)$ and $(a^2 - ab + b^2)$ are prime.

Example 1. — Multiply $(2\,x - y)(4\,x^2 + 2\,xy + y^2)$.

Solution. — Use formula 1 with $a = 2\,x$ and $b = y$:

$$(2\,x - y)(4\,x^2 + 2\,xy + y^2) = (2\,x)^3 - y^3 = 8\,x^3 - y^3.$$

Example 2. — Factor: $27\,x^3 - 8\,y^3$.

Solution. — Use formula 1 with $a = 3\,x$ and $b = 2\,y$:

$$(27\,x^3 - 8\,y^3) = (3\,x - 2\,y)(9\,x^2 + 6\,xy + 4\,y^2).$$

EXERCISE 9

Multiply by inspection:

1. $(a - x)(a^2 + ax + x^2)$.
2. $(2\,b + w)(4\,b^2 - 2\,bw + w^2)$.
3. $[(x + y) + z][(x + y)^2 - z(x + y) + z^2]$.
4. $[k - (y - x)][k^2 + k(y - x) + (y - x)^2]$.

Factor:

5. $x^3 - 8$.
6. $c^3 + 64\,d^3$.
7. $27\,y^3 - 8$.
8. $125\,r^3 - 216\,a^3$.
9. $y^3 + 8\,a^3$.
10. $(a + w)^3 - 27$.
11. $x^3 - 1$.
12. $3\,y^3 + 24$.
13. $a^3 + \frac{1}{8}$.
14. $64\,m^3 + \frac{1}{27}\,n^3$.
15. $\frac{1}{8}\,x^3 - \frac{1}{27}\,y^3$.
16. $(x+y)^3 - (y+z)^3$.

MISCELLANEOUS PROBLEMS ON FACTORING

Factor by any method:

1. $ax^2 - 2ax - 35a$.

2. $x^4 + 4x^2 + 16$.

3. $4y^2 - z^2 + 4xy + x^2$.

4. $y^3 - 2y^2 + y - 2$.

5. $ax^3 + bx^2 - ac^2x - bc^2$.

6. $16c^2 - 20rd - 4d^2 - 25r^2$.

7. $10c^2 + 19ac + 6a^2$.

8. $24a^3 - 20a^2 - 5 + 6a$.

9. $40x^3 - 42 + 35x^2 - 48x$.

10. $2a - b - 2ac + bc$.

11. $w^6 - 7w^3 - 8$.

12. $8y + 49x^2 - y^2 - 16$.

13. $36 - 12ab - a^2 - 36b^2$.

14. $20c^2 - 30cd - 140d^2$.

15. $8 + 27w^3$.

16. $x^2 - m^2 - 2my - y^2$.

17. $x^2 - y^2 + 6x + 9$.

18. $x^4 - 16y^4$.

19. $3z^2 + 7 + 22z$.

20. $4r^2s^2 + 4rs^2 - 15s^2$.

21. $9c^4 - 34c^2d^2 + 25d^4$.

22. $b^6 - b^4 - b^2 + 1$.

23. $-6x^2 + ax + bx + a^2 + 2ab + b^2$.

24. $81z^2 + 20w - 4w^2 - 25$.

25. $18x^2y^2 - 3xy^2 - 10y^2$.

26. $x^4 - y^4 + 2xy(x^2 - y^2)$.

27. $c^4 + 4d^4$.

28. $x^3 - 27y^3$.

29. $4a^2y^2 - 28ay^2 + 49y^2$.

30. $t^2 - r^2 + s^2 - 2st$.

31. $9m^4 + 2m^2t^2 + t^4$.

32. $6abx^2 - b^2x^2 + 6aby^2 - b^2y^2$.

33. $4c^2d^2 - 8cdx - 21x^2$.

34. $\frac{1}{25}a^4 - \frac{9}{49}c^2$.

35. $.25x^2 - .3xy + .09y^2$.

36. $x^4 + 64$.

37. $15ay - 6by + 5az - 2bz$.

38. $169m^2 - 26mn + n^2$.

39. $4z^3 - 10z^2 + 15 - 6z$.

40. $6.25z^2 - 10zw + 4w^2$.

41. $y^2 - 9x^2 + 25 + 6xz - 10y - z^2$.

42. $x^2 + 4z^2 + 4xz - 4xy + 4y^2 - 8yz$.

43. $2rx + 2rz - sx + 3sy - 6ry - sz$.

44. $10xy - 8xw + 15cy + 4bw - 12cw - 5by$.

45. $2x^3 - x^2y + 2x^2 - xy + 2x - y$.

46. By direct multiplication show that

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3; \quad (1)$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3. \quad (2)$$

Expand by use of the formulas of Problem 46:

47. $(a + b)^3$. **48.** $(c - y)^3$. **49.** $(3 + a)^3$. **50.** $(5 - c)^3$. **51.** $(3x + 2)^3$. **52.** $(20 + a)^3$. **53.** $(2 - \frac{1}{2}x)^3$. **54.** $(\frac{1}{3} - y)^3$.

CHAPTER III

FRACTIONS

17. Fundamental principle. — *The numerator and the denominator of a fraction may be* ***multiplied or divided by the same number*** *(**not 0**) without changing the value of the fraction.*

Thus, $\frac{3}{7} = \frac{3 \times 4}{7 \times 4} = \frac{12}{28}$; $\frac{24}{36} = \frac{24 \div 12}{36 \div 12} = \frac{2}{3}$.

In particular, *the value of a fraction is not changed if both numerator and denominator are multiplied by* -1.

Thus, $\frac{3}{5} = \frac{-3}{-5}$; $\frac{x-y}{c-d} = \frac{y-x}{d-c}$.

NOTE. — When no sign is written at the left of a fraction, the sign is understood to be plus. Thus, $\frac{3}{5}$ means $+\frac{3}{5}$.

18. To reduce a fraction to its lowest terms,

a. *find the prime factors of numerator and denominator, and*

b. *divide both numerator and denominator by their common factors.*

Example 1. — Reduce to lowest terms: $\frac{9x^2 - 16y^2}{3x^2 + 2xy - 8y^2}$.

Solution. — 1. $\frac{9x^2 - 16y^2}{3x^2 + 2xy - 8y^2} = \frac{(3x - 4y)(3x + 4y)}{(3x - 4y)(x + 2y)}$

2. $= \frac{3x + 4y}{x + 2y}$.

NOTE. — In Example 1, the division of the numerator and the denominator by $(3x - 4y)$ was indicated by *canceling* $(3x - 4y)$.

EXERCISE 10

Multiply both numerator and denominator:

1. Of $\frac{4}{5}$ by 7.
2. Of $\frac{x}{7\,b}$ by 3.
3. Of $\frac{3\,x-y}{x+y}$ by $(3\,x+y)$.
4. Of $\frac{x+y}{c+d}$ by $(x-y)$.
5. Of $\frac{x-y}{c+d}$ by -3.
6. Of $\frac{4\,x-5}{7}$ by -2.

Reduce to lowest terms:

7. $\frac{15}{36}$.
8. $\frac{39}{65}$.
9. $\frac{3\,a}{7\,a}$.
10. $\frac{5(3+d)}{6(3+d)}$.
11. $\frac{5\,x^3y^4}{3\,xy^5}$.
12. $\frac{a^3b^2}{a^4y}$.
13. $\frac{12\,x^3}{15\,x^4}$.
14. $\frac{bd(x+y)}{b(x+y)}$.
15. $\frac{12\,ab^3(x+y)}{4\,b^2(x+y)}$.
16. $\frac{(c+d)(x^2-y^2)}{(c+d)(y+z)}$.
17. $\frac{(3+4\,d)(x-y)}{(4+5\,d)(x-y)}$.
18. $\frac{x^2-y^2}{x-y}$.
19. $\frac{m^2+m-56}{m^2-m-42}$.
20. $\frac{y^2-9\,y+18}{2\,y^2+2\,y-24}$.
21. $\frac{a-2\,b}{a^2-4\,b^2}$.
22. $\frac{a^2-11\,ab+28\,b^2}{a^2-14\,ab+49\,b^2}$.
23. $\frac{3\,am^2-3\,an^2}{3\,m^2+6\,mn+3\,n^2}$.
24. $\frac{cr^2-cr-12\,c}{3\,r^2+13\,r+12}$.
25. $\frac{a^2b-8\,ab+12\,b}{ca^2-12\,ac+36\,c}$.
26. $\frac{2\,x^5+3\,x^4+2\,x+3}{4\,x^2-9}$.
27. $\frac{a^3-b^3}{a^2-b^2}$.
28. $\frac{4\,x^2-12\,xy+9\,y^2}{8\,x^3-27\,y^3}$.
29. $\frac{2\,y^2+12\,yz-80\,z^2}{2\,y^2-32\,z^2}$.
30. $\frac{9-r^4}{12+2\,r^2-2\,r^4}$.
31. $\frac{9\,x^4-49\,x^2y^2}{28\,x^2y^2-12\,x^3y}$.
32. $\frac{a^3+b^3}{a^2-2\,ab-3\,b^2}$.
33. $\frac{a^2+11\,ab+28\,b^2}{a^2+14\,ab+49\,b^2}$.
34. $\frac{x^3-8}{x^3-2\,x^2+4\,x-8}$.
35. $\frac{2\,a^2-ab-6\,b^2}{4\,b^2+4\,ab-3\,a^2}$.

19. Change in the sign before a fraction. — If the numerator or the denominator of a fraction is multiplied by -1, the sign before the fraction must be changed to keep the fraction unaltered in value, because, by the laws of signs,

$$\frac{a}{b} = -\left(\frac{-a}{b}\right); \quad \frac{a}{b} = -\left(\frac{a}{-b}\right).$$

NOTE. — Recall that, if each of an odd number of factors in a product is multiplied by -1, the product is multiplied by -1. If each of an even number of factors is multiplied by -1, the product is unchanged in value. Thus,

$$(x-y)(c-d) = (y-x)(d-c);$$
$$(x-y)(-3b)(-d) = -(y-x)(3b)(d).$$

Example 1. — Reduce to lowest terms: $\dfrac{x^2-9}{12+2x-2x^2}$.

Solution. — 1. $\dfrac{x^2-9}{12+2x-2x^2} = \dfrac{(x-3)(x+3)}{2(3-x)(2+x)}$

2. $= -\dfrac{(x-3)(x+3)}{2(x-3)(2+x)} = -\dfrac{(x+3)}{2(2+x)}$.

In Step 2, the $(3-x)$ in the denominator in Step 1 was multiplied by -1, and hence the sign before the fraction was changed.

EXERCISE 11

Write equivalent fractions in lowest terms, having no minus signs in numerator or denominator:

1. $\dfrac{2}{-3}$.
2. $-\dfrac{-b}{2b}$.
3. $\dfrac{c}{-4c}$.
4. $\dfrac{2(-3x)(-5x)}{-4x}$.
5. $-\dfrac{-21cd(5d)}{5cd(-7c^2)}$.
6. $\dfrac{-2(-12m)}{-84m}$.
7. $\dfrac{5b(-6c)}{-2c(-15b)}$.
8. $\dfrac{-8y(-x^2)}{3x(-y)}$.
9. $\dfrac{-x(-3y)}{(-y)^3}$.

Reduce to lowest terms:

10. $\dfrac{ax - 5a}{5 - x}$.

11. $\dfrac{(c - d)^2}{cd - c^2}$.

12. $\dfrac{m^2 - n^2}{n^2 + mn - 2m^2}$.

13. $\dfrac{r^4 - 9}{-2r^4 + 2r^2 + 12}$.

14. $\dfrac{4ay - 3ax}{3x^2 + 2xy - 8y^2}$.

15. $\dfrac{6 + t - t^2}{t^2 + t - 12}$.

20. Lowest common multiple. — The **degree** of an integral rational term in certain letters is the *sum of the exponents with which these letters appear in the term.* The degree of an integral rational polynomial is *the degree of its term of highest degree.*

Thus, x^4yz is of the fourth degree in x, of the first degree in y, and of the sixth degree in x, y, and z; $(3x^3 + c^2)$ is of the third degree in x and c.

The **lowest common multiple** (**L. C. M.**) of two or more integral rational expressions, with integral coefficients, is the expression of lowest degree, and with the smallest integral coefficients, which contains each of the given expressions as a factor.

To find the L. C. M. of two or more expressions:

1. *Find the prime factors of each of the expressions.*

2. *Select all of the different prime factors and give to each the highest exponent with which it appears in any expression.*

3. *Form the product of the factors selected in Step 2, leaving the result in factored form.*

Example 1. — Find the L. C. M. of $(18 - 2x^2)$, $(3x^2 - 18x + 27)$, and $(4x^2 - 16x + 12)$.

Solution. — 1. $18 - 2x^2 = 2(3 - x)(3 + x) = -2(x - 3)(x + 3)$.

2. $3x^2 - 18x + 27 = 3(x - 3)(x - 3) = 3(x - 3)^2$.

3. $4x^2 - 16x + 12 = 2 \cdot 2(x - 3)(x - 1) = 2^2(x - 3)(x - 1)$.

4. The prime factors are 2, 3, $(x - 3)$, $(x + 3)$, and $(x - 1)$.

5. Therefore, L. C. M. $= 2^2 \cdot 3 \cdot (x - 3)^2(x + 3)(x - 1)$.

Check. — The L. C. M. should be exactly divisible by each of the given expressions: We verify this by mental division. Thus,

$$(\text{L. C. M.}) \div (18 - 2x^2) = \frac{2^2 \cdot 3 \cdot (x-3)^2(x-1)(x+3)}{-2(x-3)(x+3)}$$
$$= -2 \cdot 3(x-3)(x-1).$$

NOTE. — In Example 1, $(3 - x)$, of Step 1, and $(x - 3)$, of Step 2, were not distinct prime factors because $(3 - x) = -(x - 3)$.

EXERCISE 12

Find the L. C. M. *of the expressions and check:*

1. 15 and 6.
2. 8 and 28.
3. $6ab$ and $7a^2b$.
4. $9x^2y$ and $6x^3y$.
5. $24y^3$ and $18yz^2$.
6. $18y^4$ and $33y^2z$.
7. $(4x - 12)$ and $(3x - 9)$.
8. $(4y - 8)$ and $(y^2 + y - 6)$.
9. $(y - 5)$ and $(2y^2 - 50)$.
10. $(4x^2 - 16)$ and $(x - 2)^2$.
11. $(x - 2)^3$ and $(x^2 + x - 6)$.
12. $(2y^2 + 4y - 30)$ and $(y^2 - 9)$.
13. $(y^2 - 25)$ and $(5 - y)^2$.
14. $(6 - x - x^2)$ and $(x^2 - 9)$.
15. $x^2 + 2x - 24$; $x^2 - x - 42$; $x^2 - 49$.
16. $8y^3 - x^3$; $2y^2 + 5xy - 3x^2$.
17. $3x^3 - 3y^3$; $2x^2 + 4xy - 6y^2$; $x^2 - y^2$.
18. $2a - 4$; $5a + 15$; $3a^2 + 3a - 18$.
19. $2a + 10b$; $4a^2 + 2ab$; $a^2 + 10ab + 25b^2$.
20. $2y^2 - y - 3$; $3y^2 + 7y + 4$; $6y^2 - y - 12$.
21. $y^2 - 5y + 6$; $y^2 + 2y - 8$; $y^2 + y - 12$.

21. **The lowest common denominator** (L. C. D.) of two or more fractions is the L. C. M. of their denominators.

Rule. — To change two or more fractions to equal fractions having their L. C. D.:

1. *Find the* L. C. D. *of the fractions.*
2. *For each fraction, divide the* L. C. D. *by the denominator of the fraction; multiply both numerator and denominator by this quotient.*

Example 1. — Change to equal fractions having their L. C. D.:

$$\frac{4x}{x^2-9} \text{ and } \frac{3x}{x^2-5x+6}.$$

Solution. — 1. The denominators are

$x^2-9=(x-3)(x+3)$; $x^2-5x+6=(x-2)(x-3)$.

2. The L. C. D. is $(x-3)(x+3)(x-2)$.

3. For the first fraction, (L. C. D.) $\div(x^2-9)=(x-2)$.

4. Hence, $\frac{4x}{x^2-9}=\frac{4x(x-2)}{(x^2-9)(x-2)}=\frac{4x(x-2)}{(x-3)(x+3)(x-2)}$.

5. For the second fraction, (L. C. D.) $\div(x^2-5x+6)=(x+3)$.

6. Hence, $\frac{3x}{x^2-5x+6}=\frac{3x(x+3)}{(x-3)(x+3)(x-2)}$.

22. Addition of fractions. — Indicated additions and subtractions of fractions can be conveniently performed only when the fractions have a common denominator.

The sum and the difference of two fractions with a common denominator:

1. *Their sum is a fraction whose numerator is the sum of their numerators and whose denominator is the common denominator.*

2. *Their difference is a fraction whose numerator is the difference of their numerators, and whose denominator is the common denominator.*

Thus, $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$; $\frac{3}{4}-\frac{5+x}{4}=\frac{3-(5+x)}{4}$.

Rule. — To perform indicated additions and subtractions of fractions:

1. *Find their* L. C. D. *and, by the Rule of Section* 21, *change each fraction to an equal fraction whose denominator is the* L. C. D.

2. *The denominator of the final result is the* L. C. D.

3. *The numerator of the result is obtained by combining the numerators of the fractions obtained in Step* 1, *placing each numerator in parentheses preceded by the sign of its fraction.*

4. *Remove parentheses in the numerator found in Step* 3 *and reduce the resulting fraction to its lowest terms.*

Illustration. $\dfrac{2x}{16x^2-8x+1}-\dfrac{x^2}{16x^2-1}$

$$=\frac{2x}{(4x-1)(4x-1)}-\frac{x^2}{(4x-1)(4x+1)}$$

$$=\frac{2x(4x+1)}{(4x-1)(4x-1)(4x+1)}-\frac{x^2(4x-1)}{(4x-1)(4x-1)(4x+1)}$$

$$=\frac{2x(4x+1)-x^2(4x-1)}{(4x-1)^2(4x+1)}=\frac{9x^2+2x-4x^3}{(4x-1)^2(4x+1)}.$$

Check. — When $x=1$, $\dfrac{9x^2+2x-4x^3}{(4x-1)^2(4x+1)}$ becomes $\dfrac{7}{9\cdot 5}$;

$\dfrac{2x}{16x^2-8x+1}-\dfrac{x^2}{16x^2-1}$ becomes $\dfrac{2}{9}-\dfrac{1}{15}=\dfrac{10-3}{9\cdot 5}=\dfrac{7}{9\cdot 5}$.

NOTE. — In checking the addition of fractions by substitution, do not select zero values of the letters or values which make some of the denominators zero.

NOTE. — Recall that any integral rational expression may be written as a fraction whose denominator is 1. Thus, $5=\dfrac{5}{1}$; $a+b=\dfrac{a+b}{1}$.

EXERCISE 13

Reduce to a common denominator and combine into a single fraction; check by substitution of convenient values for the letters:

1. $\dfrac{5x+3}{7}-\dfrac{3x-1}{5}-3.$

2. $\dfrac{4y-5}{12}-\dfrac{2a-7}{16}+4.$

3. $\dfrac{9a-5b}{4ab}-\dfrac{5a-6b}{3a^2b}.$

4. $\dfrac{3x}{b}+\dfrac{y}{c}-1.$

5. $\dfrac{3a}{4c}-5a-\dfrac{6c}{a}+c.$

6. $\dfrac{2+y^2}{2+y}-\dfrac{2-y^2}{y-2}.$

7. $\dfrac{1}{x+3}-\dfrac{4x-1}{x-2}.$

8. $\dfrac{5x}{x-3}-\dfrac{4x^2+3x-1}{x^2+x-12}.$

9. $\dfrac{a-1}{a+1}-\dfrac{a+1}{a-1}+\dfrac{4a^2+1}{a^2-1}.$

10. $\dfrac{x-1}{2x^2+5x+3}-\dfrac{x+1}{2x+3}.$

11. $\dfrac{2a-n}{2a-2n}-\dfrac{3a-4n}{3a-3n}.$

12. $3a-\dfrac{3-7a}{2a-3}+1.$

13. $a-4-\dfrac{2+11a}{3a}.$

14. $\dfrac{5a+7}{4x}-\dfrac{5a-1}{4x+1}.$

15. $\dfrac{a}{4m - m^2} + \dfrac{2}{m^2 - 16}$.

16. $\dfrac{3x}{x^2 - 9} + \dfrac{5}{3 - x}$.

17. $\dfrac{2a}{a^2 + 4a - 60} - \dfrac{2}{a + 10}$.

18. $\dfrac{n - 6}{n^2 + 3n - 4} - \dfrac{1}{1 - n}$.

19. $\dfrac{x}{x^2 + xy + y^2} - \dfrac{yx^2}{x^3 - y^3}$.

20. $\dfrac{1}{y^2 - xy} + \dfrac{1}{xy - x^2}$.

21. $1 - \dfrac{6}{x} - \dfrac{1}{x^2 - 5x}$.

22. $a + b - \dfrac{a^3 - b^3}{a^2 - ab + b^2}$.

23. $\dfrac{r - a}{r^2 - 6ar + 9a^2} - \dfrac{a}{r^2 + 4ar - 21a^2}$.

24. $\dfrac{a + 1}{a^2 - a - 6} - \dfrac{a - 4}{a^2 - 4a + 3} + \dfrac{a + 4}{a^2 + a - 2}$.

25. $\dfrac{3a + 2}{6a^2 - a - 1} + \dfrac{a + 3}{3a^2 + 7a + 2} - \dfrac{a - 2}{2a^2 + 3a - 2}$.

23. Multiplication and division of fractions.

Rule 1. — The product of two or more fractions is a fraction in which the numerator is the product of the numerators of the given fractions, and the denominator is the product of their denominators.

Thus, $\dfrac{3}{4} \cdot \dfrac{5}{11} = \dfrac{15}{44}$; $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$.

In multiplying two fractions proceed as follows:

1. *Find the prime factors of their numerators and denominators.*

2. *Form the product, dividing out (canceling) common factors of numerator and denominator.*

3. *Leave the numerator and denominator of the result in factored form unless otherwise directed.*

Illustration. $\dfrac{2x^2 + 7x - 15}{2x^2 - 3x - 14} \cdot \dfrac{2x^2 - 19x + 42}{8x - 12}$

$= \dfrac{(2x - 3)(x + 5)}{(2x - 7)(x + 2)} \cdot \dfrac{(2x - 7)(x - 6)}{4(2x - 3)} = \dfrac{(x + 5)(x - 6)}{4(x + 2)}$.

Check. — When $x = 1$, the original product becomes

$-\dfrac{6}{15} \cdot \dfrac{25}{4} = -\dfrac{5}{2}$; the final result becomes $\dfrac{(6)(-5)}{4(3)} = -\dfrac{5}{2}$.

Rule 2. — To divide one fraction by another:

a. *Invert the divisor fraction.*

b. *Multiply the dividend by the inverted divisor.*

Illustration. $$\frac{\dfrac{y^3 - xy^2}{x^3 + x^2y}}{\dfrac{x^2 - 2xy + y^2}{x^2 - xy - 2y^2}} = \frac{y^3 - xy^2}{x^3 + x^2y} \cdot \frac{x^2 - xy - 2y^2}{x^2 - 2xy + y^2}$$

$$= -\frac{y^2(x - y)}{x^2(x + y)} \cdot \frac{(x - 2y)(x + y)}{(x - y)(x - y)} = -\frac{y^2(x - 2y)}{x^2(x - y)}.$$

Since $a = \dfrac{a}{1}$, it follows that

$$a\left(\frac{c}{b}\right) = \frac{a}{1} \cdot \frac{c}{b} = \frac{ac}{b}; \quad \frac{c}{b} \div a = \frac{c}{b} \div \frac{a}{1} = \frac{c}{b} \cdot \frac{1}{a} = \frac{c}{ab}.$$

Hence, **to multiply a fraction by a number, multiply the numerator by the number; to divide a fraction by a number, multiply the denominator by the number.**

EXERCISE 14

Perform the indicated operations and, if letters are involved, check the result by substitution of convenient values for the letters:

1. $8\left(\dfrac{17}{6}\right)$.

2. $-3\left(\dfrac{14}{9}\right)$.

3. $\dfrac{3}{8} \div 5$.

4. $\dfrac{5}{3} \div 2$.

5. $c^2\left(\dfrac{7a}{15}\right)$.

6. $\dfrac{5h^2}{b} \div b$.

7. $\dfrac{5a}{7b} \cdot \dfrac{3a^2}{8b^2}$.

8. $\dfrac{15}{3} \div \dfrac{30}{7}$.

9. $\dfrac{16a}{b} \div \dfrac{32a^2}{15b^2}$.

10. $\dfrac{x^2 - y^2}{x^2 + 2xy + y^2} \cdot \dfrac{2x + 2y}{3x}$.

11. $\dfrac{3t}{y^2 - 6y + 8} \div \dfrac{2t^3}{y^2 - y - 12}$.

12. $\dfrac{4y^2 - 1}{y^3 - 16y} \cdot \dfrac{y^2 + 4y}{1 - 2y}$.

13. $\dfrac{(r + 2s)^2}{r - s} \div \dfrac{rs + 2s^2}{r^2 - rs}$.

14. $\dfrac{16x - 4}{5x - 5} \cdot \dfrac{20x + 5}{6x + 6} \cdot \dfrac{x^2 + 2x + 1}{16x^2 - 1}$.

15. $\dfrac{4m^2 + 8m + 3}{2m^2 - 5m + 3} \cdot \dfrac{6m^2 - 9m}{4m^2 - 1}$.

16. $\dfrac{x^3+8y^3}{x^3-8y^3}\cdot\dfrac{x-2y}{x+2y}\cdot\dfrac{x^2+2xy+4y^2}{x^2-2xy}$.

17. $\dfrac{\dfrac{a^2-ab-2b^2}{a^3-9ab^2}}{\dfrac{a-2b}{a-3b}}$.

18. $\dfrac{\dfrac{8n^3+1}{2n^2+4n}}{\dfrac{4n^2-2n+1}{n^2+4n+4}}$.

19. $\dfrac{\dfrac{2a^2-ab-3b^2}{9a^2-25b^2}}{\dfrac{3a^2+ab-2b^2}{9a^2-30ab+25b^2}}$.

20. $\dfrac{\dfrac{3w^2-5wz+2z^2}{16w^2-8wz+z^2}}{\dfrac{27w^3-8z^3}{4w^3-w^2z}}$.

24. Mixed expressions. — A mixed expression is one consisting of an integral rational part and of one or more fractions.

Thus, $\left(3x-5+\dfrac{4}{x-1}\right)$ is a mixed expression.

If a mixed expression appears in a product or in a quotient, reduce the expression to a simple fraction as in Section 22, before performing other operations.

A **complex fraction** is one in which one or more fractions appear either in the numerator or in the denominator, or in both numerator and denominator.

To reduce a complex fraction to a simple fraction, *first reduce the numerator and the denominator to simple fractions and then find the quotient of these simple fractions.*

Illustration.
$$\frac{x+\dfrac{4}{x-3}}{3-\dfrac{12}{x^2-9}}=\frac{\dfrac{x(x-3)+4}{x-3}}{\dfrac{3(x^2-9)-12}{x^2-9}}=\frac{\dfrac{x^2-3x+4}{x-3}}{\dfrac{3x^2-39}{(x-3)(x+3)}}$$
$$=\frac{x^2-3x+4}{x-3}\cdot\frac{(x-3)(x+3)}{3x^2-39}=\frac{(x+3)(x^2-3x+4)}{3(x^2-13)}.$$

Check. — When $x=1$, the original fraction becomes

$$\frac{1+\dfrac{4}{-2}}{3-\dfrac{12}{-8}}=\frac{1-2}{3+\dfrac{3}{2}}=-\frac{1}{\dfrac{9}{2}}=-\frac{2}{9}.$$

The final result becomes $\dfrac{4(2)}{3(-12)}=-\dfrac{2}{9}$.

EXERCISE 15

Reduce to a simple fraction in lowest terms and check by substitution of values for the letters:

1. $\dfrac{4 - \frac{5}{3}}{1 + \frac{2}{5}}$.

2. $\dfrac{15 - \frac{2}{6}}{\frac{1}{3} + \frac{1}{5}}$.

3. $\dfrac{17}{1 + \frac{5}{3}}$.

4. $\dfrac{1 - \frac{9}{11}}{3}$.

5. $\dfrac{\frac{2}{x}}{3 - \frac{1}{x}}$.

6. $\dfrac{1 + \frac{1}{3a}}{9a^2 - 1}$.

7. $\dfrac{y - \frac{1}{2}}{4y^2 - 1}$.

8. $\dfrac{x + 3a}{\frac{x^2}{9} - a^2}$.

9. $\dfrac{3}{\frac{5x}{b} - \frac{x^2}{2b}}$.

10. $\dfrac{\frac{2a}{a - b} - 1}{\frac{a}{a - b} - 1}$.

11. $\dfrac{\frac{3}{x} + \frac{5}{y}}{\frac{3}{x^2} - \frac{6}{y^2}}$.

12. $\dfrac{m + \frac{8}{m^2}}{12m}$.

13. $\dfrac{x - \frac{2d^2}{x + d}}{x - \frac{3xd - d^2}{x + d}}$.

14. $\dfrac{4 - \frac{1}{y}}{2 - \frac{2x - 1}{x^2}}$.

15. $\dfrac{x + \frac{y^2}{x - 2y}}{\frac{y}{x^2 - 4y^2}}$.

16. $\dfrac{\frac{2x}{r - x} - \frac{1}{x}}{\frac{r}{3x} - \frac{3x}{r}}$.

17. $\dfrac{a + \frac{b^2}{a - b}}{a + \frac{2ab + b^2}{a - b}}$.

18. $\dfrac{\frac{5}{7} - \frac{x}{x - 2}}{\frac{1}{x^2 - 3x + 2}}$.

19. $\left(1 + \frac{2a}{7}\right)\left(a - \frac{1}{7 + 2a}\right)$.

20. $\left(a + \frac{b^2}{a}\right)\left(1 - \frac{b}{a^2 + b^2}\right)$.

21. $\left(\frac{a}{9b} - \frac{b}{a}\right)\left(\frac{6a^2b^2}{a + 3b}\right)$.

22. $\left(4 - \frac{1}{y + 1}\right) \div \left(16 + \frac{7}{y^2 - 1}\right)$.

23. $\left(\frac{5x + 2}{x^2 - 1} + 6\right) \div \left(2 + \frac{1}{x - 1}\right)$.

24. $\left(1 + \frac{4}{5x}\right) \div (5x^2 - x - 4)$.

The **reciprocal** of a number is the *quotient of* 1 *divided by the number.*

Thus, the reciprocal of 3 is $\frac{1}{3}$; of $\frac{a}{b}$ is $\frac{1}{\frac{a}{b}}$ or $\frac{b}{a}$.

Find the reciprocals of the following expressions:

25. 75. **26.** $\frac{3}{5}$. **27.** $\frac{8}{3}$. **28.** -4. **29.** $27\,b$.

30. $\frac{a}{6\,b}$. **31.** $\frac{5\,c}{3\,d}$. **32.** $10\frac{1}{3}$. **33.** $\frac{4\,x-7}{3\,x+3}$. **34.** $\left(4-\frac{5}{x}\right)$.

Simplify to a simple fraction in lowest terms:

35. $\dfrac{2\,a^2}{5\,a-\dfrac{4\,a-1}{1-\dfrac{2\,a+5}{3\,a-2}}}$.

36. $\dfrac{1-\dfrac{1}{x}}{x-\dfrac{1}{2\,x-\dfrac{x+1}{x}}}$.

CHAPTER IV

LINEAR EQUATIONS IN ONE UNKNOWN

25. Equations. — An **equation** is a statement that two number expressions are equal. The two expressions are called **sides** or **members** of the equation. An equation in which the two members are equal for all values of the letters involved is called an **identical equation**, or, for short, an **identity**. An equation in which the two members are not equal for all values of the letters involved is called a **conditional equation.**

Thus, $(a - b)^2 = a^2 - 2ab + b^2$ is an identity; $x - 2 = 0$ is a conditional equation because the two members are equal only when $x = 2$.

The word *equation* by itself will be used in referring to both identities and conditional equations, except where such usage would cause confusion. Usually, however, the word equation refers to a conditional equation. At times, to emphasize the fact that some equation is an identity, we shall use " $\equiv$ " instead of " $=$ " between the members.

A conditional equation may be thought of as an interrogative sentence, asking for the values of the letters which make the two members equal. These letters, whose values are asked for, are called **unknowns**. Some of the letters in an equation may represent known numbers.

Thus, $x^2 + 3x - 5 = 0$ is an equation in the unknown x.

26. An equation is satisfied by a set of values of the unknowns if the equation becomes an identity when these values are substituted for the unknowns. A **solution** of an equation is a set of values of the unknowns which satisfies

the equation. A solution of an equation involving a single unknown is called a **root** of the equation.

Thus, $x = 3$ is a solution, or root, of $2x - 3 = 3$.

To solve an equation means to find all of its solutions.

27. Equivalent equations. — Two equations are **equivalent** if they have the same solutions.

Thus, $x - 3 = 0$ and $3x - 9 = 0$ are equivalent; they have the same root, $x = 3$.

Each of the following operations on an equation yields an equivalent equation *:

1. *Addition of the same number to both members.*
2. *Subtraction of the same number from both members.*
3. *Multiplication or division of both members by the same number, provided that this number is not zero and* ***does not involve the unknowns*** *of the equation.*

28. Mechanical processes grow out of the preceding operations.

A. *A term appearing on both sides of an equation can be* **cancelled**, *by subtracting the term from both members.*

Illustration. $x + a = 5 + a$.
Subtract a from both sides: $x = 5$.

B. *A term can be* **transposed** *from one member to the other, with the sign of the term changed, by subtracting it from both members.*

Illustration. $x + a - 5 = 7$.
Subtract $(a - 5)$ from both sides: $x = 7 - a + 5$.

C. *The signs of all terms on both sides can be changed, by multiplying both sides by* -1.

Illustration. $3x - b = 5 - ax$.
Multiply both sides by -1: $-3x + b = -5 + ax$.

* We shall assume these facts as true without giving their proofs.

29. Linear equations. — An **integral rational** equation is one in which each member is an integral rational polynomial in the unknowns. A **linear equation**, or **an equation of the first degree**, is an integral rational equation in which the terms involving the unknowns are of the 1st degree.

Thus, $3x - 5 = 7$ is a linear equation in x.

30. To solve a linear equation in one unknown:

1. *Clear of fractions by multiplying both sides by the* L. C. D. *of the denominators in the equation; remove all parentheses.*

2. *Transpose all terms involving the unknown to the left member and all other terms to the right member. Combine terms in the unknown, exhibiting it as a factor.*

3. *Divide both sides by the coefficient of the unknown.*

4. *Check the solution by substitution in the original equation.*

NOTE 1. — When convenient, the explanations of solutions of equations will be abbreviated by using the letters **A**, **S**, **D**, and **M** in the manner illustrated below.

"**A**: (**8**)." means *add* 8 *to both sides of the previous equation.*

"**S**: (**16** x)." means *subtract* 16 x *from both sides of the previous equation.*

"**M**: (**− 2**)." means *multiply both sides of the previous equation by* − 2.

"**D**: (7)." means *divide both sides of the previous equation by* 7.

Example 1. — Solve: $\frac{x-4}{3} - \frac{x-3}{2} = \frac{3+x}{10} - 2.$

Solution. — 1. **M**: (30). $10(x-4) - 15(x-3) = 3(3+x) - 60.$

2. $10x - 40 - 15x + 45 = 9 + 3x - 60.$

3. **S**: $(3x + 5).$ $-5x + 5 = 3x - 51.$

4. $-8x = -56.$

5. **D**: $(-8).$ $x = 7.$

Check. — Substitute $x = 7$ in the original equation:

Does $\frac{7-4}{3} - \frac{7-3}{2} = \frac{3+7}{10} - 2$? Does $1 - 2 = 1 - 2$? Yes.

Example 2. — Solve for x: $b(b + x) = a^2 - ax.$

Solution. — 1. $b^2 + bx = a^2 - ax.$

2. **A**: $(ax - b^2).$ $ax + bx = a^2 - b^2.$

3. Factor. $x(a+b)=(a-b)(a+b)$.

4. D: $(a+b)$. $x=a-b$.

Check. — Substitute $x=(a-b)$ in the original equation:

Does $b(b+a-b)=a^2-a(a-b)$? Does $ab=a^2-a^2+ab$? Yes.

EXERCISE 16

Solve the following equations:

1. $5x-3=x+5$.
2. $3x-6=14-x$.
3. $\frac{x}{3}-\frac{1}{4}=\frac{1}{2}x-\frac{5}{4}$.
4. $\frac{3-2x}{3}+\frac{1}{5}=2-\frac{x-3}{5}$.
5. $5(x-3)=3(x-4)$.
6. $2t-5=3-4t$.
7. $z+\frac{2}{3}=3z+5$.
8. $5x-4(x-3)=\frac{x-1}{2}$.
9. $(x-1)(x+1)=(x+2)(x-3)$.
10. $5(x^2-2)=(5x-1)(x+3)$.
11. $\frac{1}{2}(x-1)+\frac{1}{3}-\frac{1}{6}x=0$.
12. $4y-\frac{23}{20}=\frac{6}{5}y+\frac{1}{2}y$.
13. $\frac{5}{4}z-\frac{3}{2}z=\frac{1}{8}-\frac{1}{3}z$.
14. $.5x-1.3=.2x+4.1$.
15. $.96x-.46=.79+.21x$.
16. $\frac{5}{6}x=\frac{7}{4}x-\frac{3}{8}x+\frac{13}{6}$.
17. $3x-.53=.03x+.361$.
18. $(x-1)^2=(x+1)^2$.
19. $.05x-.708x-1.82=-.504$.
20. $(2y-3)(2y+3)=4y^2-5y$.

Solve for x, or for s, and check the solution:

21. $bx=3+a$.
22. $3.15=x+.05x$.
23. $x(a-b)=c+d$.
24. $16x=4+h$.
25. $k-px=0$.
26. $34+5cx=0$.
27. $as+bs=5$.
28. $cx-dx=3b$.
29. $5x-dx=-4$.
30. $s-5a=4a+2s$.
31. $c-2x=3x-4d$.
32. $5x+4=a-bx$.
33. $3x+4bx-2=0$.
34. $4bs-5as=16b^2-25a^2$.
35. $ax-a^2=x-1$.
36. $y(x+y)=c(c-x)$.
37. $m(bs-2m)=b(2m-bs)$.
38. $y(x-y+2d)=d(x+d)$.

39. $4\,x(c + d) = 1 + 3\,cx + 3\,dx$.

40. $3(2\,x - 4)(x + 7) = 2(3\,x - 2)(x + 5) - 3(x - 2)$.

41. Solve $s = vt$ for t.

42. Solve $s = \frac{1}{2}\,gt^2$ for g.

43. Solve $K = \frac{1}{2}\,mv^2$ for m.

44. Solve $t^2 = \dfrac{\pi l}{g}$ for l.

45. Solve $A = P(1 + rt)$, (*a*) for P; (*b*) for r; (*c*) for t.

46. Solve $S = k + vt$, (*a*) for v; (*b*) for t.

31. Operations which may not yield equivalent equations.

A. *If both members of an equation are multiplied by an expression* ***involving the unknowns****, the new equation may have more solutions than the original equation.*

Illustration 1. $(x - 3) = 0$ has only one root, $x = 3$. On multiplying both sides of $(x - 3) = 0$ by $(x + 2)$, we obtain $(x^2 - x - 6) = 0$. By substitution, we verify that this equation has two roots, $x = 3$ and $x = -2$. The latter root was introduced by the multiplication.

B. *If both members of an equation are divided by an expression* ***involving the unknowns****, the new equation may have fewer roots than the original equation.*

Illustration 2. By substitution we verify that $x = 1$ and $x = 2$ are roots of $x^2 - 3\,x + 2 = 0$. On dividing both members by $(x - 2)$, we obtain $(x - 1) = 0$, whose only root is $x = 1$. The root $x = 2$ was lost by the division.

Example 1. — Solve: $\dfrac{x}{x^2 - 1} - \dfrac{1}{x^2 - 1} + \dfrac{2}{x + 1} = 0$.

Solution. — **M**: $(x^2 - 1)$. $\quad x - 1 + 2(x - 1) = 0$;

$3\,x = 3$; $x = 1$.

Check. — Since $x = 1$ makes $(x^2 - 1) = 0$ in the denominators of the original equation, hence $x = 1$ cannot be admitted as a root because division by zero is not allowed. Hence, the original equation has no roots.

A value of the unknown, such as $x = -2$ in Illustration 1, which satisfies a derived equation but does not satisfy the original one, is called an **extraneous root**. Whenever an

operation of type A, or of type B, is used in solving, test all values obtained, in order to reject extraneous roots, if any.

NOTE. — The following demonstration of the absurd result that $2 = 1$ illustrates the contradictions that would arise if division by zero were allowed.

1. Suppose that	$y = b.$
2. **M**: (y).	$y^2 = by.$
3. **S**: (b^2).	$y^2 - b^2 = by - b^2.$
4. Factor.	$(y - b)(y + b) = b(y - b).$
5. **D**: $(y - b)$.	$y + b = b.$
6. Since $y = b$ (Step 1),	$b + b = b$, or $2\,b = b.$
7. Dividing both sides by b, we obtain	$2 = 1.$

Discussion. — In Step 5, we divided by zero, because, since $y = b$, then $y - b = 0$. Hence, no equation beyond Step 4 is valid, because division by zero is not allowed.

Example 2. — Solve: $\dfrac{27}{z-5} - \dfrac{8}{z+2} = \dfrac{18}{z^2 - 3z - 10}.$

Solution. — 1. The L. C. D. of the denominators is $(z - 5)(z + 2)$. Multiply both sides of the original equation by this L. C. D.:

$$27(z + 2) - 8(z - 5) = 18.$$

2. Expand and collect: $94 + 19z = 18$; $19z = -76$; $z = -4$.

Check. — Does $\dfrac{27}{-4-5} - \dfrac{8}{-4+2} = \dfrac{18}{16+12-10}$?

Does $-3 + 4 = \frac{18}{18}$? Yes.

EXERCISE 17

Each equation, if cleared of fractions properly, will reduce to a linear equation. This reduction may be prevented, and extraneous roots may be introduced, if an incorrect L. C. D. of the fractions is used.

Solve:

1. $\dfrac{1}{3} - 5 + \dfrac{2}{x} = \dfrac{6}{2x}.$

2. $\dfrac{2}{3x} - \dfrac{3}{x} + \dfrac{5}{2x} = 1 - \dfrac{11}{6x}.$

3. $\dfrac{21a^2 + 7a + 11}{7a^2 - 4a - 9} = 3.$

4. $\dfrac{2x}{5} + 2 = x - \dfrac{2}{5}.$

5. $2 = \dfrac{5x + 7}{2(x - 1)}.$

6. $\dfrac{5}{6} - \dfrac{21}{5x} = \dfrac{5x - 2}{10x}.$

7. $\frac{1}{5t} - \frac{3}{10t} - \frac{2}{15t} = -\frac{7}{12}$.

8. $\frac{7x}{x+3} + \frac{5x}{x-1} = \frac{12(x^2-1)}{x^2+2x-3}$.

9. $\frac{12s-5}{21} - \frac{3s+4}{3(3s+1)} = \frac{4s-5}{7}$.

10. $\frac{t+3}{t+1} - \frac{t}{t-2} = \frac{8t+20}{t^2-t-2}$.

11. $\frac{3}{3t-4} = \frac{4t+5}{6t-1} - \frac{2t}{3t-4}$.

12. $\frac{3(a+1)}{2a-4} - \frac{4}{5a-10} = \frac{1}{2}$.

13. $\frac{3x+1}{x-3} - 1 = \frac{2x}{3-x}$.

14. $\frac{4+3x}{2x+5} = \frac{3x+2}{2x+3}$.

15. $\frac{2x-5}{3x-1} + \frac{3}{2} = 0$.

16. $\frac{x-1}{x+9} = 1 - \frac{5}{x+2}$.

17. $\frac{3}{x-3} - \frac{4}{x-4} = \frac{7}{4(4-x)}$.

18. $\frac{4}{x-4} - \frac{1}{x-1} = \frac{15}{x-1}$.

19. $\frac{2-x}{3-x} - \frac{x+2}{x+3} = \frac{5}{x^2-9}$.

20. $\frac{3x-1}{1.5} - \frac{x^2-.64}{..5(x+1)} = \frac{11}{15}$.

Solve for x, or for w:

21. $\frac{cx-c^2}{b+x} + b = 0$.

22. $\frac{a-w}{5bw} = \frac{10b-a}{15aw} - \frac{1}{3a}$.

23. $\frac{3x}{2x+n} + \frac{x+2n}{2x} = 2$.

24. $\frac{w}{b-1} + \frac{w}{b+1} = 12$.

25. $\frac{a+x}{a-x} = \frac{a+b}{a-b}$.

26. $\frac{x}{.3+x} - .6 = \frac{.3-x}{.3+x}$.

27. $\frac{x+a}{x-2a} - \frac{x-a}{x+3a} + \frac{2ax-19a^2}{x^2+ax-6a^2} = 0$.

28. $\frac{w(a+4c)-b^2}{a^2-b^2} + \frac{w-b}{a+b} = \frac{w+a}{a-b}$.

29. $\frac{x^2+3}{2(x^3-8)} + \frac{1}{6(x-2)} = \frac{2x-1}{3(x^2+2x+4)}$.

30. $\frac{5}{2x+5m} - \frac{2}{3x-4m} = \frac{3m}{6x^2+7mx-20m^2}$.

31. In $C = \dfrac{Kab}{b - a}$, (1) solve for a; (2) solve for b.

32. In $l = \dfrac{l - M}{Mt}$, (1) solve for M; (2) solve for t.

33. In $w = \dfrac{5\,a}{1 + \dfrac{l^2}{600\,d^2}}$, solve for a.

34. In Problem 33, by substitution find the value of w correct to two decimal places when $a = 18$, $l = 12$, and $d = 10$.

32. Applications. — In applications of algebra, number expressions described in words must be translated into algebraic expressions.

Example 1. — The length of a rectangular lot is three times its width. If the length is decreased by 20 feet and the width increased by 10 feet, the area is increased by 200 square feet. Find the original dimensions of the lot.

Solution. — 1. Let x feet be the width; then $3\,x$ feet is the length; the area is $x(3\,x)$, or $3\,x^2$ square feet.

2. When the dimensions are changed, the length is $(3\,x - 20)$ feet, the width is $(x + 10)$ feet; the area is $(3\,x - 20)(x + 10)$ square feet.

3. Since the new area is 200 square feet larger than the original area,

$$3\,x^2 + 200 = (3\,x - 20)(x + 10).$$

4. $3\,x^2 + 200 = 3\,x^2 + 10\,x - 200$; $10\,x = 400$; $x = 40$.

5. The original width is 40 feet, and the length is 120 feet.

Check by substituting in the problem. — If the width is 40 ft. and the length 120 ft., the area is $40(120) = 4800$ square feet. The length, decreased by 20, is 100 ft.; the width, increased by 10, is 50 ft.; the new area is $50(100)$, or 5000 square feet. Since $5000 = 4800 + 200$, the results are correct.

Note. — The solution of a written problem should always be checked by substituting the solution in the *original written statement.*

EXERCISE 18

Solve each problem by using only one unknown member:

1. The perimeter of a certain rectangle is 330 feet and the altitude is four sevenths of the base. Find the dimensions.

2. There are two consecutive positive integers such that the sum of twice the larger and four times the smaller is 110. Find these integers.

HINT. — Let x be the smaller integer; the other is $(x + 1)$.

3. A is now four times as old as B. In 6 years, A will be twice as old as B will be then. How old is each now?

4. Find three consecutive integers whose sum is 51.

5. A sum of money amounting to \$3.80 consists of pennies, nickels, and quarters. The number of nickels is 3 times the number of pennies, and the number of quarters is 3 less than the number of nickels. How many of each coin are there?

HINT. — If there are p pennies, the value of the nickels is $5 \cdot (3\ p)$ ¢.

6. A sum of money amounting to \$5.35 consists of nickels, dimes, and quarters. There are 5 more quarters than there are nickels; the number of dimes is 1 less than three times the number of nickels. How many of each kind of coin are there?

PROBLEMS INVOLVING MIXTURES

7. A grocer has some tea worth 90¢ per pound and some worth 40¢. How many pounds of each are used in forming 100 pounds of a mixture worth 60¢ per pound?

HINT. — Let n be the number of pounds of 90¢ tea used. Then, the cost of the 40¢ tea used is $40(100 - n)$ cents. The total cost is 100(60) cents.

8. A grocer wishes to form a mixture of 60 pounds of coffee, worth 40¢ per pound, by mixing 55¢ coffee and 30¢ coffee. How many pounds of each kind should he use?

9. How many gallons of a solution containing 80% alcohol should be added to 5 gallons of a 20% solution, to give a mixture containing 30% alcohol?

HINT. — 1. Let x be the number of gal. to be added.
2. In x gal., 80% pure, there are $.80\ x$ gal. of alcohol.
3. In $(5 + x)$ gal., 30% pure, there are $.30(5 + x)$ gal. of alcohol.
4. The alcohol in the final mixture is the sum of the alcohol in the x gal. and in the 5 gal.

10. How many ounces of pure silver must be added to 150 ounces, 40% pure, to give a mixture 60% pure?

11. How many pounds of metal, 50% gold, must be added to 200 pounds, 30% gold, to give a mixture 40% gold?

12. How many pounds of cream, containing 30% butter fat, should be added to 500 pounds of milk containing 2% butter fat, to give milk containing 3.5% butter fat?

WORK PROBLEMS

13. If A can plow a certain field in 8 hours, and if B can do it in 5 hours, how long will it take for them to plow the field if they work together?

HINT. — 1. Let x hours be the time it will take, when both work.

2. A plows $\frac{1}{8}$ of the field in 1 hour, and hence $\frac{x}{8}$ of it in x hours.

3. Since A and B together complete the work in x hours, the sum of the fractional parts of the work they do in x hr. must equal 1.

14. How long will it take for A and B, working together, to do a piece of work which A alone could do in 10 days, and B alone in 20 days?

15. How long will it take for A, B, and C, working together to paint a house if, working alone, A can do it in 10 days, B in 12 days, and C in 15 days?

16. In a certain factory, 1000 articles of a given type can be turned out by a first machine in 8 hours, by a second machine in 16 hours, and by a third in 20 hours. How long will it take to turn out 1000 articles if all three machines work at the same time?

PROBLEMS CONCERNING FRACTIONS

17. If $(37 - x)$ is divided by x, the quotient is 3 and the remainder is 5. Find the value of x.

HINT. — See formulas 1 and 2, Section 9.

18. Separate 54 into two parts such that the larger part divided by the smaller gives 3 as a quotient and 10 as a remainder.

19. What number, when added to both numerator and denominator of $\frac{3}{7}$, will cause the fraction to equal $\frac{3}{4}$?

20. What number, when subtracted from both numerator and denominator of $\frac{3}{4}$, will cause the fraction to equal $\frac{1}{2}$?

PROBLEMS INVOLVING UNIFORM MOTION

NOTE. — In uniform motion, the speed, or velocity is constant. That is, *equal distances are passed over in equal intervals of time.* The **rate** or **velocity** of motion is the distance traveled *in one unit of time.* If d is the distance traveled in t units of time, and if r is the rate, then $\boldsymbol{d = rt}$.

21. At what rate is a man traveling who goes 300 miles (*a*) in 10 hours; (*b*) in h hours?

22. How long does it take an automobile to go 150 miles if its rate is (*a*) 20 miles per hour; (*b*) x miles per hour?

23. A and B travel toward each other from points 300 miles apart, A at the rate of 20 miles per hour, and B at 25 miles per hour. When will they meet if they start at the same time?

HINT. — Let h be the number of hours they travel. In h hours, A travels $20h$ miles. The sum of the distances they travel is 300 miles.

24. Two men traveling in opposite directions at the rates of 18 miles and 22 miles per hour, respectively, started at the same time from the same place. After how many hours will they be 250 miles apart?

25. A and B started toward each other at the same time from points 240 miles apart, and met in 5 hours. If A traveled 12 miles per hour more than B, what were their rates?

26. At how many minutes after 2 P.M. will the minute hand of a clock overtake the hour hand?

Solution. — 1. Let x be the number of minutes. In x minutes, the minute hand goes x minute spaces, and the hour hand $\frac{x}{12}$ minute spaces.

2. The minute hand gains $\left(x - \frac{x}{12}\right)$ spaces on the hour hand in x minutes. Since the minute hand starts 10 minute spaces behind the hour hand, the gain in x minutes must equal 10 spaces, or $x - \frac{x}{12} = 10$; solving, we obtain $x = 10\frac{10}{11}$.

27. How long after 3 o'clock will the minute hand of a clock overtake the hour hand?

28. Two automobiles started from the same place in opposite directions. The first traveled 5 miles more per hour than the second. At the end of 8 hours they were 360 miles apart. What were their rates of travel?

29. A freight train leaves Chicago for New Orleans at a speed of 25 miles per hour. Six hours later an express train leaves Chicago for New Orleans at a speed of 55 miles per hour. When will the express train overtake the freight train?

RIVER PROBLEMS

NOTE 1. — When a boat is going downstream on a river, the rate of the boat is equal to the rate of the current of the river, plus the rate which the boat would have if the water were still. Thus, in a river whose current is 3 miles per hour, a man who can row 4 miles per hour in still water could row 7 miles per hour downstream, and (4 − 3) or 1 mile per hour upstream.

30. A motor boat, which can go 10 miles per hour in still water, travels on a river where the rate of the current is 3 miles per hour. (*a*) What will be the rate of the boat going downstream? (*b*) How long will it take the boat to go 28 miles upstream?

31. Some men in a boat found that it took them as long to go 8 miles downstream on the river, of problem 30, as it did to go 3 miles upstream. At what rate would their boat travel in still water?

HINT. — If x miles is the rate of the boat in still water, the time going downstream is $\frac{8}{x+3}$ hours.

32. One river flows at the rate of 2 miles per hour and another flows at the rate of 4 miles per hour. A man finds that he can row 16 miles downstream on the second river in the time he takes to row 12 miles downstream on the first river. At what rate can he row in still water?

PROBLEMS INVOLVING SIMPLE INTEREST

NOTE. — If r is the rate of interest expressed as a decimal, the interest I earned by a principal P in t years is given by $I = Prt$. The sum of the principal and the interest is called the amount. If A represents the amount after t years, then $A = P + Prt$, or $A = P(1 + rt)$.

33. State the following rates in decimal form: 4%; $3\frac{1}{2}\%$; $2\frac{1}{4}\%$.

34. State the following rates as per cents: .045; .065; .07.

35. By use of $I = Prt$, find the interest for $\frac{1}{2}$ year on \$750, at (*a*) 6%; (*b*) 7%; (*c*) 5%; (*d*) 3%.

36. (*a*) Find the rate r, if $P = \$1000$, $I = \$250$, and $t = 3$ years. (*b*) Find the principal P, if $A = \$4000$, $r = .05$, and $t = 3$ years.

37. One sum is invested at 6%, and a second at 7%. The total invested is \$2000 and the total interest after 1 year is \$135. Find the sums invested.

38. One sum is invested at 6% and twice as much at 8%. The total interest after 1 year is \$220. Find the sums invested.

39. \$3000 of Mr. B's income is not taxed. All of his income over \$3000 is taxed 2%, and all income above \$8000 is taxed 2% in addition. If B pays a tax of \$320, what is his income?

HINT. — On an income of \$15,000, the tax would be

$$.02(15000 - 3000) + .02(15000 - 8000) = 240 + 140 = 380.$$

40. A man invests \$1500 at 5% and \$2000 at 6%. How much more must he invest at 7% so that his annual income will be 6.5% of the total sum invested?

PROBLEMS INVOLVING SIMPLE LEVERS

NOTE. — A lever consists of a rigid rod with one point of support called the fulcrum. If a weight w is attached to the lever at a certain point, the distance of w from the fulcrum is called the **lever arm** of w. A familiar instance of a lever is a teeter board.

FIG. 1.

In Physics it is proved that the following statement is true, if two or more weights are placed along a lever in such a way that the lever is in a position of

equilibrium. *If each weight is multiplied by its lever arm, the sum of these products, for all weights on one side of the fulcrum, equals the sum of the products for all weights on the other side.*

Thus, in Figure 1, the sum of the products for the weights at the left is $5(80) + 4(250) = 1400$, and for those at the right the sum is $4(100) + 5(200) = 1400$; hence this lever is balanced.

41. A weight of 300 pounds is placed on a lever 10 feet from the fulcrum. How far from the fulcrum on the other side must a weight of 250 pounds be placed in order to give equilibrium?

42. A and B together weigh 350 pounds. They balance when A is seated 4 feet from the fulcrum on one side of a lever and B is seated 3 feet from the fulcrum on the other side. How much does each weigh?

43. A boy weighing 90 pounds sits 6 feet from the fulcrum on a teeter board. Another boy weighing 70 pounds sits 5 feet from the fulcrum, on the other side. To balance the board, where should a third boy sit, if he weighs 80 pounds?

44. Two girls weighing 75 pounds and 85 pounds, respectively, wish to sit at the ends of a teeter board 10 feet long. How far from the lighter girl should the fulcrum be placed in order that the girls may balance?

45. How heavy a weight can a man lift with a lever 8 feet long, if the fulcrum is placed 1 foot from the weight and if the man exerts a force of 150 pounds?

HINT. — The force he exerts is equivalent to placing a weight of 150 pounds at his end of the bar. He can lift as much as this weight would balance.

EXERCISE 19

REVIEW OF CHAPTERS I TO IV

Perform indicated operations and simplify:

1. $(3x - y)(3x + y) - 3(x - y) - (x - y)(x + y)$.

2. $(8t - 4) - [(5t + 1) + (3t - \{2t - 1\})]$.

3. $(12z - 9w)(12z + 9w)$.

4. $(m^2 - 4n)(m^2 + 4n)$.

5. $(3z + h)^2$.

6. $(5c - 3d)^2$.

7. $(3x - 5)(2x + 4)$.

8. $(4p - 7r)(7p + 3r)$.

9. $(z - 2)^3$.

10. $(\frac{3}{5}x - \frac{2}{3}y)(\frac{3}{5}x + \frac{2}{3}y)$.

Factor:

11. $-2ax-6ay+9az$.

12. y^2-49x^2.

13. $8z^2-22z+15$.

14. $18w^2-3w-10$.

15. $25x^2-10x+1$.

16. $36a^2b^2-\frac{1}{4}$.

17. $4a^3+15-6a-10a^2$.

18. $c^3+2c^2-9c-18$.

19. $x^4+2x^2y^2+9y^4$.

20. $x^4-2x^2y^2+y^4$.

21. $20z^2+zw-w^2$.

22. $8x^3-y^3$.

23. $(m-n)^2-8(m-n)+15$.

24. $(x+4)^2-y^2$.

25. $4a^2+4a+1-36w^2$.

26. $21z^2+2zx-8x^2$.

27. $49r^4-1-2x-x^2$.

28. $a^2+4ab-c^2+4b^2$.

29. b^2+6b-c^2+9.

30. $c^2d^2-10cd-96$.

Perform indicated operations and simplify:

31. $5\left(\dfrac{3x-8}{2}\right)$.

32. $\dfrac{\frac{7x}{x-3}}{2}$.

33. $\dfrac{5}{\frac{3x-2}{4}}$.

34. $\dfrac{3}{x-y}+\dfrac{2y+x}{y^2-x^2}$.

35. $3m-\dfrac{4}{m}-\dfrac{9m^2-16}{3m+1}$.

36. $\dfrac{4z^2+1}{4z^2-1}-\dfrac{2z-1}{2z+1}+\dfrac{5z-3}{2z-1}$.

37. $\dfrac{2}{x^2+3x+2}-\dfrac{4}{x^2-1}+\dfrac{2}{1-x}$.

38. $\left(1+\dfrac{ab-c^2}{a^2+ab+b^2}\right)\div\left(1-\dfrac{3ab+c^2}{a^2+ab+b^2}\right)$.

39. $\dfrac{\frac{3}{x}-\frac{5}{y}}{\frac{3}{x}+\frac{4}{y}}$.

40. $\dfrac{1-\frac{2}{3a}}{a-\frac{4}{9a}}$.

41. $\dfrac{27a^2-\frac{b^3}{a}}{\frac{9a}{b}+3+\frac{b}{a}}$.

42. $\dfrac{1-\frac{2x}{y-x}}{\frac{y}{3x}-\frac{3x}{y}}$.

43. $1-\dfrac{5}{1-\frac{5}{1+a}}$.

44. $\dfrac{1-\frac{5(a^2-b^2)}{a^2-4b^2}}{1-\frac{3a+b}{a+2b}}$.

45. If $A = P(1 + rt)$, find P if $A = \$390$, $r = .06$, and $t = 5$.

Solve and check by substitution:

46. $\frac{1}{3}(x - 1) + 3 = \frac{1}{5}(x + 14)$.

47. $\dfrac{3x + 1}{3} - \dfrac{x + 2}{2x + 5} = x$.

48. $\dfrac{3w}{2w - 5} - \dfrac{3}{2} = \dfrac{7}{3w + 1}$.

49. $\frac{1}{4}(y - 2) - \frac{1}{6}(y - 4) = \frac{2}{3}$.

50. $\dfrac{1 - 5x}{2x - 3} + 3 = \dfrac{x}{2x - 3}$.

51. $\dfrac{12}{3 - x} - \dfrac{4 - x}{1 - x} = -1$.

52. $\dfrac{2x - 1}{2x - 3} + \dfrac{x^2 - x}{x^2 + 4} = 2$.

53. $\dfrac{2x + 5}{x + 7} - \dfrac{3x^2 + 24x - 6}{x^2 + 8x + 7} + 1 = 0$.

54. Is $x = -\dfrac{a + b}{2}$ a root of $\dfrac{ax}{x + b} + \dfrac{bx}{x + a} = a + b$?

Solve for x:

55. $\dfrac{x + 5a}{2} - \dfrac{x + a}{4} = 3a$.

56. $\dfrac{a + b}{x} = \dfrac{b}{x - b} + \dfrac{a}{x + a}$.

57. $\dfrac{Nn}{Mm} = \dfrac{x}{p - x}$.

58. $\dfrac{4a - x}{a - x} - 1 = \dfrac{12a}{3a - x}$.

59. How many pounds of cream, containing 40% butter fat, should be added to 200 pounds of milk, containing 3% butter fat, to give milk containing 4% butter fat?

60. A tank has one supply pipe which fills it in 5 hours and another which fills it in 9 hours. How many hours will it take to fill the tank if both pipes are used simultaneously?

61. A certain machine requires 25 hours to complete a certain job. After this machine works for 17 hours, it is decided to complete the work with a second machine which would have required 40 hours for the whole of the job. How long will it take the second machine to complete the work?

62. At how many minutes past 4 P.M. will the minute hand of a clock overtake the hour hand?

63. The denominator of a certain fraction exceeds its numerator by 5. If the numerator is decreased by 1, the resulting fraction equals $\frac{2}{5}$. Find the original fraction.

64. A boat which can travel 8 miles per hour in still water travels upstream for a certain time in a river whose current flows 2 miles per hour, and then the boat returns to its starting point. If the trip, up and back, consumed 8 hours, how long did the boat travel upstream?

65. A man makes two investments whose sum is $10,000. In one year he gained 6% interest on one investment and lost 4% of the second. His net gain was $100. Find the size of each investment.

66. A man is carrying two packages, weighing 25 and 35 pounds, respectively, by balancing them over his shoulder at the ends of a stiff rod, 4 feet long. How far from his shoulder should the 25-pound package be carried?

67. On one side of a lever, weights of 40 pounds, 45 pounds, and 55 pounds are placed 5 feet, 4 feet, and 6 feet, respectively, from the fulcrum. On the other side, a weight of 60 pounds is 6 feet from the fulcrum. How far from the fulcrum, on this side, should a weight of 70 pounds be placed, so that the lever will balance?

CHAPTER V

RECTANGULAR COÖRDINATES AND GRAPHS OF LINEAR EQUATIONS

33. The number scale. — On the scale in Figure 2, we select A to represent zero. All positive numbers are repre-

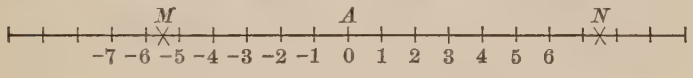


FIG. 2.

sented in order by the points to the right of A and all negative numbers by the points to the left. The unit points on the scale represent the integers and the other points represent the numbers which are not integers.

34. Greater than and less than. — We say that *a is greater than b*, or that *b is less than a*, in case $(a - b)$ is positive. We use the inequality signs $>$ and $<$ to indicate *greater than* and *less than*, respectively.

Thus, $2 > -5000$; $-8 > -36$; $-5 < 1$; $0 > -30$.

Suppose that M and N are two numbers represented on the scale in Figure 2. Then, to say that $M < N$ means, geometrically, that *M is to the left of N* in Figure 2.

35. Rectangular coördinates. — In Figure 3, $X'X$ is perpendicular to $Y'Y$. On each of these lines we lay off a scale with O as the zero on both scales. Vertical distances will be measured in terms

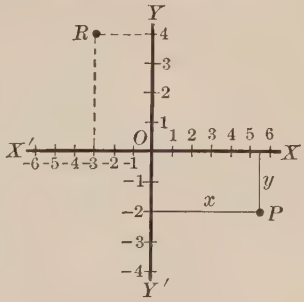


FIG. 3.

of the unit used on the $Y'Y$ scale, and horizontal distances in terms of the unit on the $X'X$ scale.

Let P be any point in the plane containing the lines $X'X$ and $Y'Y$. From P, drop perpendiculars to $X'X$ and to $Y'Y$.

The horizontal coördinate, *or the* ***x*-coordinate**, *of P is the perpendicular distance from $Y'Y$ to P, where this distance is considered positive when P is to the right, and negative when P is to the left of $Y'Y$.*

The vertical coördinate, *or the* ***y*-coördinate**, *of P is the perpendicular distance from the line $X'X$ to P, where this distance is considered positive when P is above, and negative when P is below $X'X$.*

Each of the lines $X'X$ and $Y'Y$ is called a **coördinate axis**, and their intersection is called the **origin** of the coördinate system. The horizontal coördinate of a point P is called the **abscissa** and the vertical coördinate is called the **ordinate** of P. The abscissa and ordinate, together, are called the **rectangular coördinates** of P.

In Figure 3, the coördinates of P are $x = 5.5$ and $y = -2$. The coördinates of a point are usually written together within parentheses, with the abscissa first. Thus, in Figure 3, we say that P is the point $(5.5, -2)$, meaning that the abscissa of P is 5.5 and the ordinate is -2.

To each point P there corresponds one and only one pair of coördinates, and to each pair of coördinates there corresponds one and only one point.

To plot a point whose coördinates are given means to locate the point and to mark it with a dot or a cross.

Example 1. — Plot the point $(-3, 4)$.

Solution. — At $x = -3$ on $X'X$, in Figure 3, erect a perpendicular to $X'X$. Go up 4 vertical units on this perpendicular to reach the point R for which $x = -3$ and $y = 4$.

Second solution. — At $+4$ on $Y'Y$ erect a perpendicular to $Y'Y$. Go 3 horizontal units to the left on this perpendicular to reach $(-3, 4)$.

EXERCISE 20

1. Insert the proper sign, $<$ or $>$, between the numbers of each pair: -5 and -2; -23 and -2; 0 and -12; 3 and -13.

2. Set up a system of coördinates on a piece of cross-section paper and plot the following points: $(3, 7)$; $(-5, -6)$; $(3, -6)$; $(-4, 4\frac{1}{2})$; $(2\frac{1}{2}, 4\frac{1}{4})$; $(0, 6)$; $(-3, 0)$; $(0, 0)$.

3. A square, with its sides parallel to the axes, has one corner at $(-1, -2)$ in a coördinate system where the units on the axes are the same. If each side of the square is 4 units long, find the coördinates of its other corners if the square lies above and to the right of $(-1, -2)$.

4. Three corners of a rectangle are $(-1, 2)$, $(2, 2)$, and $(2, 5)$; find the coördinates of the other corner.

5. In what line is every point (*a*) whose abscissa is 5; (*b*) whose ordinate is -3?

6. Where does a point lie if (*a*) both of its coördinates are positive; (*b*) both of its coördinates are negative; (*c*) its abscissa is positive and its ordinate negative?

7. In the following table, the two numbers in each column are the coördinates of a point. Plot all of the points and join them by a smooth curve:

$x =$	-3	$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$y =$	10	$7\frac{1}{4}$	5	$3\frac{1}{4}$	2	$\frac{5}{4}$	1	$\frac{5}{4}$	2	$3\frac{1}{4}$	5	$7\frac{1}{4}$	10

36. Constants and variables. — In a given problem, a **constant** is a number symbol whose value does not change during the discussion. A **variable** is a number symbol which may take on different values during the discussion.

Illustration. The area A of a circle is given by the formula $A = \pi r^2$, where r is the radius. In considering the areas of different circles, π is a constant (3.1416) for all circles, but r and A are variables because the radii and areas are different for different circles. Moreover, in $A = \pi r^2$, r and A are related variables.

37. Solutions of an equation in two variables. — If an equation involves two variables x and y, then x and y are related variables. If a value is assigned to x, we can compute the corresponding value of y from the equation; or, if a value is assigned to y, we can compute the corresponding value of x.

Thus, let x and y be two variables related by the equation $y = 2x + 5$. If the value $x = 3$ is assigned to x, then $y = 2(3) + 5$, or $y = 11$; hence, $(x = 3, y = 11)$ is a pair of corresponding values of the related variables x and y. If $y = 7$ is assigned as the value of y, then $7 = 2x + 5$; $2 = 2x$, or $x = 1$. Hence, $(x = 1, y = 7)$ is a pair of corresponding values of the variables x and y.

A **solution** of an equation involving two variables x and y is a **pair of corresponding values** of the variables x and y which satisfy the equation.

Thus, $(x = 3, y = 11)$ and $(x = 1, y = 7)$ are two solutions of $y = 2x + 5$.

38. Functions. — *If a first variable and a second variable are so related that, whenever a value is assigned to the first variable, a corresponding value of the second variable can be determined, the second variable is called a function of the first variable.* The first variable is called the **independent variable** and the second variable is called the **dependent variable.**

Thus, in $A = \pi r^2$, the area of the circle is a function of r, the radius.

To say that a variable y is a function of the variable x means that *the value of y depends on the value of x;* if a value is assigned to the independent variable x, then a corresponding value of y can be determined.

39. In case x and y are related by an equation, as for instance $y = 2x + 5$, then y is a function of x, and, likewise, x may be considered as a function of y. For, if the value of either variable is assigned, the value of the other variable can be computed from the equation.

40. Plotting solutions of an equation.

Example 1. — Plot the solutions of $3x - 5y = 15$. (1)

Solution. — 1. Place $x = -5$; $-15 - 5y = 15$; $-5y = 30$, or $y = -6$. Hence, $(x = -5, y = -6)$ is a solution of (1). Place $y = 0$ in (1): $3x = 15$; $x = 5$. Hence, $(x = 5, y = 0)$ is a solution of (1). The two solutions just obtained and others found similarly are tabulated in columns in the table:

$x =$	-5	-2	0	3	5	6
$y =$	-6	$-4\frac{1}{5}$	-3	$-1\frac{1}{5}$	0	$\frac{3}{5}$

2. Each solution in the table can be taken as the coördinates of a point in a coördinate system. In Figure 4, the solutions $(-5, -6)$, $(-2, -4\frac{1}{5})$, etc., are plotted. It is seen that all of these points lie on a straight line.

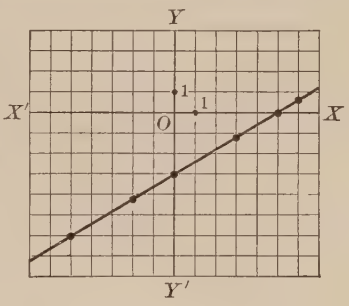


FIG. 4.

NOTE. — Example 1 makes it evident that $3x - 5y = 15$ has *infinitely many solutions*, and that we can obtain as many as we desire by the method of Example 1. Usually, it is true that *any* equation in two variables has infinitely many solutions.

41. The graph, or the **locus of an equation** involving two variables x and y, is the curve (or curves) on the (x, y) coördinate plane which has the following properties:

A. *The graph contains every point whose coördinates (x, y) form a solution of the equation.*

B. *If a point is on the graph, then the coördinates of the point form a solution of the equation.*

In geometrical language, the graph of an equation can be described as the **locus** *of all points whose coördinates form a solution of the equation.*

42. To graph an equation means to draw its graph.

In a more advanced course in mathematics, it is proved that, if a, b, and c are any constants, *the graph of the linear equation $ax + by = c$ is a straight line.*

Thus, the heavy line in Figure 4 is the graph of $3x - 5y = 15$.

To graph a linear equation, proceed as follows:

1. *Find three solutions of the equation.*

2. *Take each solution as the coördinates of a point; plot these points, and join them by a straight line.*

NOTE. — If the three points do not fall on a straight line, there is an error in at least one of the three solutions which were used.

EXERCISE 21

1. Find a solution of $3x + 5y = 18$, (*a*) by substituting $x = 2$; (*b*) by substituting $y = 4$; (*c*) by substituting $y = 0$.

2. Graph the equation $3x + 5y = 18$. From the graph, read the solution in which (*a*) the value of x is -3; (*b*) the value of y is -2.

Graph the following equations:

3. $x - 5y = 10$.
4. $4x = 7 + 2y$.
5. $x + 4y = 0$.
6. $5x = 8 - y$.
7. $3y - 2x = 9$.
8. $3x = 28$.
9. $-x - 2y = 8$.
10. $y = 7$.
11. $5y + 33 = 0$.

HINT for Problem 8. — A pair of values (x, y) satisfies $3x = 28$, if y has any value and $x = 9\frac{1}{3}$. The graph is perpendicular to the x-axis.

12. The relation between a temperature expressed in degrees Fahrenheit, represented by F, and in degrees Centigrade, represented by C, is given by $F = \frac{9}{5}C + 32$. Draw a graph of this equation from $C = -20$ to $C = 100$. From this graph, (*a*) read the value of F when $C = 10$; $C = -10$; $C = 50$; (*b*) read the value of C when $F = 40$; $F = -20$; $F = 150$.

13. Where does the graph of $x - y = 7$ cut (*a*) the x-axis; (*b*) the y-axis?

14. Without drawing the graph of $4x - 7y = 21$, find where it would cut (*a*) the x-axis; (*b*) the y-axis.

15. Find the locus of $6x + 5y = 30$.

CHAPTER VI

SIMULTANEOUS LINEAR EQUATIONS IN TWO AND IN THREE UNKNOWNS

43. Simultaneous equations. — A **solution** of a system of two equations in two unknowns, x and y, is a pair of corresponding values of x and y which satisfy both equations. **To solve** a system means to find all of its solutions. If a system has a solution, the two equations are called **simultaneous.**

Example 1. — Solve graphically: $\begin{cases} x - y = 5, & (1) \\ x + 2y = 2. & (2) \end{cases}$

Solution. — 1. In Figure 5, the line AB is the graph of (1) and CD is the graph of (2).

2. Line AB contains every point whose coördinates form a solution of (1); CD contains every point whose coördinates form a solution of (2). Hence, the intersection of AB and CD is the only point whose coördinates form a solution of both equations. The point of intersection, E, is $(4, -1)$. Hence, $(x = 4, y = -1)$ is the solution of the system.

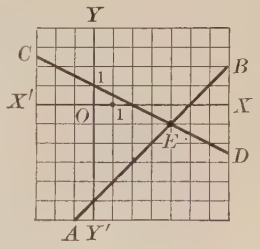


FIG. 5.

Check. — The student can verify that $(4, -1)$ is a solution by substituting $(x = 4, y = -1)$ in (1) and (2).

NOTE. — Usually, a graphical method will give only an approximate solution because in obtaining it we estimate the values of certain coördinates by measuring line segments in the graph.

To solve a system of two simultaneous linear equations in two unknowns graphically, proceed as follows:

1. *On one coördinate system draw the graph of each equation.*

2. *Measure the coördinates of the point of intersection of the graphs; these coördinates form the solution.*

Usually, a system of two linear equations in two unknowns has one and only one solution, as was the case in Example 1.

44. Special cases in solving a system of two linear equations in two unknowns:

A. *If the graphs of the equations are parallel lines, the system has no solution, and the equations are called* **inconsistent** *equations.*

B. *If the graphs of the equations are the same straight line, every solution of either equation is also a solution of the other; the equations are then said to be* **dependent** *equations. In this case we can find as many solutions of the system as we desire, by finding solutions of either of the equations.*

EXERCISE 22

Solve each system graphically. If there is one and only one solution, check your result by substitution.

1. $\begin{cases} 2x + y = 3, \\ x - y = 3. \end{cases}$

2. $\begin{cases} 3x + y = 10, \\ x + 2y = 5. \end{cases}$

3. $\begin{cases} u - y = 4, \\ 2u + y = 11. \end{cases}$

4. $\begin{cases} 2x + 4y = 7, \\ 2x - y = 2. \end{cases}$

5. $\begin{cases} x - 3y = 6, \\ 2x - 6y = 0. \end{cases}$

6. $\begin{cases} x = 3, \\ 2x + y = 8. \end{cases}$

7. $\begin{cases} y = -2, \\ x - y = 4. \end{cases}$

8. $\begin{cases} 5a + 2b = 10, \\ a - b = 9. \end{cases}$

9. $\begin{cases} x - y = 2, \\ 2x + y = 5. \end{cases}$

10. $\begin{cases} x + y - 8 = 0, \\ 2x + 2y - 3 = 0. \end{cases}$

11. $\begin{cases} y + 2 - x = 0, \\ x + 4 = 0. \end{cases}$

12. $\begin{cases} 3x - y = 5, \\ 6x - 2y = 10. \end{cases}$

13. (*a*) Draw the graph of $x + y = 3$. (*b*) Multiply both sides of the equation by 2 and draw the graph of the resulting equation.

14. Judging from the results of problem 13, how are two linear equations related when they have the same graph?

45. Solution by elimination by substitution.

Example 1. — Solve for x and y: $\begin{cases} 4x + 5y = 6, & (1) \\ 2x + 3y = 4. & (2) \end{cases}$

Solution. — 1. Solve (2) for x: $2x = 4 - 3y$; $x = \frac{4 - 3y}{2}$. (3)

2. In (1), substitute $x = \frac{4 - 3y}{2}$: $4\left(\frac{4 - 3y}{2}\right) + 5y = 6$. (4)

3. Solve (4) for y: $8 - 6y + 5y = 6$; $y = 2$.

4. Substitute $y = 2$ in (3): $x = \frac{4 - 6}{2}$; $x = -1$.

5. The solution of the system is $(x = -1, y = 2)$.

Check. — Does $\begin{cases} 4(-1) + 5(2) = 6 \\ 2(-1) + 3(2) = 4 \end{cases}$? Does $\begin{cases} -4 + 10 = 6 \\ -2 + 6 = 4 \end{cases}$? Yes.

NOTE. — In obtaining equation 4 in the solution of Example 1, we say that we have **eliminated** x between equations 1 and 2.

To solve two linear equations in two unknowns by elimination by substitution, proceed as follows:

1. *Solve one equation for one unknown in terms of the other.*

2. *Substitute the resulting expression for the first unknown in the other equation.*

3. *Solve the equation obtained in Step 2 for the second unknown.*

4. *Substitute the value of the second unknown in any equation involving both unknowns, and solve for the first unknown.*

46. Elimination by addition or subtraction.

Example 1. — Solve for x and y: $\begin{cases} 3x + 3y = 5, & (1) \\ 2x - 5y = 7. & (2) \end{cases}$

Solution. — 1. Multiply (1) by 2: $6x + 6y = 10$. (3)

2. Multiply (2) by 3: $6x - 15y = 21$. (4)

3. Subtract, (3) − (4): $21y = -11$; $y = -\frac{11}{21}$. (5)

4. Substitute $y = -\frac{11}{21}$ in (1): $3x - 3(\frac{11}{21}) = 5$; $x = \frac{46}{21}$.

5. The solution is $(x = \frac{46}{21}, y = -\frac{11}{21})$.

To solve two linear equations in two unknowns by elimination by addition or subtraction, proceed as follows:

1. *In each equation, multiply (if necessary) both members by a properly chosen number, so as to obtain two equations in which the coefficients of one of the unknowns have the same absolute value.*

2. *If these coefficients have the same sign, subtract each member of one equation from the corresponding member of the other; if the coefficients have opposite signs, add corresponding members.*

3. *Solve the equation found in Step 2 for the unknown in it.*

4. *Substitute the resulting value of this unknown in one of the original equations and solve for the value of the other unknown.*

EXERCISE 23

Solve by the substitution method and check the solution:

1. $\begin{cases} x = y + 3, \\ 2x + 3y = 11. \end{cases}$

2. $\begin{cases} y = 4 - x, \\ 3x + 4y = 4. \end{cases}$

3. $\begin{cases} x - 2y = 3, \\ 3x - 5y = 10. \end{cases}$

4. $\begin{cases} 3y - 4x = -2, \\ y - 5x = 3. \end{cases}$

5. $\begin{cases} 9z + 4y = -21, \\ 7z - 3y = 2. \end{cases}$

6. $\begin{cases} 7r + 8s = 36, \\ 11r + 5s = -4. \end{cases}$

Solve by the method of addition and subtraction and check:

7. $\begin{cases} 2x + 3y = 7, \\ 2x + y = 1. \end{cases}$

8. $\begin{cases} 3x - 4y = 14, \\ 5x + 4y = 2. \end{cases}$

9. $\begin{cases} 16m + 3n = 11, \\ 20m = 30 - 7n. \end{cases}$

10. $\begin{cases} 6r + 2t = -3, \\ 5r - 3t + 6 = 0. \end{cases}$

11. $\begin{cases} 5x + 6y = 1.97, \\ 4x - 5y = .4. \end{cases}$

12. $\begin{cases} 15c = 8 - 4d, \\ 12d = 20c - 15. \end{cases}$

Solve by any method:

13. $\begin{cases} 2x - 3y = 3, \\ 5x + 2y = 17. \end{cases}$

14. $\begin{cases} 4x + 3y = 6.4, \\ 3x - .5y = 1.5. \end{cases}$

15. $\begin{cases} 7x - 6y = 63, \\ 9x + 2y = 13. \end{cases}$

16. $\begin{cases} 3x - 4y = .5, \\ x + 2y = .8. \end{cases}$

17. $\begin{cases} 7y - 3x - 6 = 0, \\ 6x + 4y = 6. \end{cases}$

18. $\begin{cases} 8m + 9z = 7, \\ 4m - 2z + 3 = 0. \end{cases}$

19. $\begin{cases} \dfrac{3x}{2} - \dfrac{4y}{3} = -1, \\ \dfrac{2x}{3} - \dfrac{y}{4} = \dfrac{7}{12}. \end{cases}$

20. $\begin{cases} \dfrac{x+3}{4} - \dfrac{y}{2} = \dfrac{3}{2}, \\ \dfrac{x}{2} - \dfrac{y+3}{4} = \dfrac{3}{2}. \end{cases}$

21. $\begin{cases} \dfrac{3}{x} + \dfrac{5}{y} = 17, \\ \dfrac{2}{x} + \dfrac{1}{y} = 2. \end{cases}$

HINT for Problem 21. — Eliminate *without* clearing of fractions.

22. $\begin{cases} \dfrac{10}{x} - \dfrac{9}{y} = 2, \\ \dfrac{8}{x} - \dfrac{15}{y} = 1.5. \end{cases}$

24. $\begin{cases} 3x - \dfrac{z-4}{6} = 13, \\ 4x + 7 + \dfrac{2z-3}{5} = 0. \end{cases}$

23. $\begin{cases} \dfrac{w-2}{w+1} = \dfrac{z+2}{z+1}, \\ \dfrac{w-3}{w-4} = \dfrac{z+5}{z+4}. \end{cases}$

25. $\begin{cases} \dfrac{5-y+6z}{7y-9z-1} - 1 = 0, \\ \dfrac{y-3z-1}{2y-3z-1} = 1. \end{cases}$

In each problem, proceed with the solution by elimination until you recognize that the equations are inconsistent or dependent; then, graph the equations to verify your conclusion:

NOTE. — If the two equations are inconsistent or dependent, then, in eliminating one unknown, the other is also eliminated; (*a*) if the equations are dependent, an identity $0 = 0$ results; (*b*) if the equations are inconsistent, a contradictory equation, such as $0 = 36$, is obtained.

26. $\begin{cases} x = y - 3, \\ 2x - 2y = 5. \end{cases}$

29. $\begin{cases} 3x + y = 5, \\ 6x + 2y = 12. \end{cases}$

27. $\begin{cases} 2x - 3y = 5, \\ 4x - 6y = 10. \end{cases}$

30. $\begin{cases} 2x + 3y = 4, \\ 4x + 6y - 8 = 0. \end{cases}$

28. $\begin{cases} 4x - y = 7, \\ 4x = 8 + y. \end{cases}$

31. $\begin{cases} x - \frac{1}{2}y = 2, \\ 2x = y + 4. \end{cases}$

47. Simultaneous literal equations.

Example 1. — Solve for x and y: $\begin{cases} bx + y = b, & (1) \\ ax + 2y = 2a. & (2) \end{cases}$

Solution. — 1. Multiply (1) by a: $abx + ay = ab.$ (3)

2. Multiply (2) by b: $abx + 2by = 2ab.$ (4)

3. Subtract, (3) − (4): $ay - 2by = -ab;\ y(a - 2b) = -ab.$ (5)

4. Divide (5) by $(a - 2b)$: $y = \dfrac{ab}{2b - a}.$ (6)

5. If we start anew and eliminate y, as we eliminated x, we obtain

$$x = \frac{2b - 2a}{2b - a}. \tag{7}$$

Check. — Place $a = 1$ and $b = 2$ in the solution and in the original system. The solution becomes $(x = \frac{2}{3},\ y = \frac{2}{3})$; the system becomes

$$\begin{cases} 2x + y = 2, \\ x + 2y = 2. \end{cases}$$

The student can verify that this system is satisfied by $(x = \frac{2}{3},\ y = \frac{2}{3})$. This numerical check is not an absolute check but it would probably detect any error if there were one.

NOTE. — In case a system contains other letters than the ones being solved for, usually it is best to solve by finding the expression for each unknown in turn by the method of addition and subtraction. The solution can be checked by the method of the previous example. In substituting values for the literal constants, do not use values which make any denominator zero in the original system or in the solution.

EXERCISE 24

Solve for x and y or, for z and w; check when directed by the instructor:

1. $\begin{cases} 3ax + y = 1, \\ 2ax + 2y = 1. \end{cases}$

2. $\begin{cases} ax + by = 2, \\ cx + dy = f. \end{cases}$

3. $\begin{cases} aw + bz = c, \\ bw - az = c. \end{cases}$

4. $\begin{cases} \dfrac{3x}{a} + \dfrac{y}{b} = 11, \\ \dfrac{5x}{a} - \dfrac{y}{b} = 13. \end{cases}$

5. $\begin{cases} x - 2by = 5ac, \\ 2ay - 3x = 3ac. \end{cases}$

6. $\begin{cases} az - bw = 0, \\ abz + abw = (a + b)^2. \end{cases}$

7. $\begin{cases} \dfrac{x}{c} + \dfrac{y}{d} = \dfrac{1}{c} + \dfrac{1}{d}, \\ x - y = \dfrac{c - d}{cd}. \end{cases}$

8. $\begin{cases} bx + ay = a + b, \\ abx + aby = a^2 + b^2. \end{cases}$

9. $\begin{cases} \dfrac{3w}{b} + \dfrac{3z}{a} = \dfrac{2a}{b} - \dfrac{b}{a}, \\ \dfrac{w}{a} - \dfrac{z}{b} = 1. \end{cases}$

48. Systems of three linear equations in three unknowns.

Example 1. — Solve for x, y, and z: $\begin{cases} 3x + y - z = 11, & (1) \\ x + 3y - z = 13, & (2) \\ x + y - 3z = 11. & (3) \end{cases}$

Solution. — 1. Subtract, (1) − (2): $-2x - 2y = -2.$ (4)

2. Multiply (2) by 3: $3x + 9y - 3z = 39.$ (5)

3. Subtract, (5) − (3): $-2x + 8y = 28.$ (6)

4. *Solve* (4) *and* (6) *for* x *and* y:

Subtract, (6) − (4): $10y = 30;\ y = 3.$

Substitute $y = 3$ in (4): $2x - 6 = -2;\ x = 2.$

5. Substitute $(x = 2,\ y = 3)$ in (1): $6 + 3 - z = 11;\ z = -2.$

6. The solution is $(x = 2,\ y = 3,\ z = -2)$. The student should check this by substitution in (1), (2), and (3).

To solve three linear equations in three unknowns:

1. *From one pair of the equations, eliminate one of the unknowns; eliminate the same unknown from another pair of the original equations.*

2. *Solve the resulting equations for the two unknowns in them.*

3. *Substitute the values of the unknowns found in Step* 2 *in the simplest of the given equations, and solve for the third unknown.*

Usually, a system of three linear equations in three unknowns has one and only one solution, as in Example 1 of this section. In special cases, three equations in three unknowns may have no solution, in which case the equations are called inconsistent; or, a system may have infinitely many solutions.

EXERCISE 25*

Solve for the three unknowns and check by substitution:

1. $\begin{cases} x + y - 2z = 7, \\ 2x - 3y - 2z = 0, \\ x - 2y - 3z = 3. \end{cases}$

2. $\begin{cases} 2x + 3y + 2z = 2, \\ 4x - y - z = 1, \\ 2x + y + 2z = 4. \end{cases}$

3. $\begin{cases} 2s + 4y - 5t = 1, \\ s + 2y - 3t = 1, \\ s - 3t = 2. \end{cases}$

4. $\begin{cases} 6s + 3t + 2w = 1, \\ 3s - 4w = 4, \\ 5s - t = 14. \end{cases}$

5. $\begin{cases} 3x + 4y + z = -1, \\ 2x - y + 2z + 1 = 0, \\ x + 3y - z = 2. \end{cases}$

6. $\begin{cases} 2A + 3B + C = 1, \\ 4A + 3C = 1, \\ 2A - 6B - 4C = 3. \end{cases}$

*If the instructor desires, Miscellaneous Exercise 30 at the end of the chapter may be taken up immediately after completing Exercise 25.

7. $\begin{cases} x + \frac{2}{3}y = 3, \\ z + \frac{y}{2} = \frac{11}{2}, \\ 15 = \frac{5 + 6z}{x}. \end{cases}$

8. $\begin{cases} \frac{z}{3} - \frac{x + y}{2} = 2, \\ \frac{1}{2x - y} = \frac{1}{z + 5}, \\ 2x + y = 0. \end{cases}$

9. $\begin{cases} \frac{1}{x} - \frac{1}{y} - \frac{3}{z} = -10, \\ \frac{1}{x} + \frac{2}{y} - \frac{1}{z} = 2, \\ \frac{3}{x} - \frac{2}{y} + \frac{2}{z} = 5. \end{cases}$

10. $\begin{cases} \frac{1}{x} - \frac{1}{y} = 4, \\ \frac{1}{x} - \frac{1}{z} = 5, \\ \frac{1}{y} + \frac{1}{z} = 3. \end{cases}$

HINT for Problem 9. — Substitute $\frac{1}{x} = u$, $\frac{1}{y} = v$, and $\frac{1}{z} = w$. Solve for u, v, and w. From their values, determine the values of x, y, and z.

SUPPLEMENTARY MATERIAL

49. Determinants of the second order. — The adjoining symbol is called a determinant, and is an abbreviation for $(ad - bc)$. That is, by definition, $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{1}$$

The numbers a, b, c, and d are called the **elements**, and the expression $(ad - bc)$ is called the **expansion** of the determinant. This determinant is said to be of the **second order** because it has two rows and two columns of elements.

Illustration. $\begin{vmatrix} 3 & 2 \\ -4 & -5 \end{vmatrix} = 3(-5) - (-4)(2) = -15 + 8 = -7.$

EXERCISE 26

Find the expansion of each determinant:

1. $\begin{vmatrix} 2 & 1 \\ 4 & -5 \end{vmatrix}$ **2.** $\begin{vmatrix} 0 & 2 \\ 1 & -3 \end{vmatrix}$ **3.** $\begin{vmatrix} 5 & 7 \\ 8 & 2 \end{vmatrix}$ **4.** $\begin{vmatrix} a & b \\ e & f \end{vmatrix}$ **5.** $\begin{vmatrix} c & d \\ 3 & -1 \end{vmatrix}$ **6.** $\begin{vmatrix} -5 & h \\ 6 & f \end{vmatrix}$ **7.** $\begin{vmatrix} h & l \\ k & g \end{vmatrix}$ **8.** $\begin{vmatrix} a & e \\ b & k \end{vmatrix}$

50. Solution of two linear equations by determinants.

Example 1. — Solve for x and y: $\begin{cases} ax + by = e, & (1) \\ cx + dy = f. & (2) \end{cases}$

Solution. — 1. Multiply (1) by d: $adx + bdy = de.$ (3)

2. Multiply (2) by b: $bcx + bdy = bf.$ (4)

3. Subtract, (3) − (4): $(ad - bc)x = (de - bf).$ (5)

4. Suppose that a, b, c, d, e, and f are any numbers for which $(ad - bc) \neq 0$. Then, divide (5) by $(ad - bc)$:

$$x = \frac{de - bf}{ad - bc}. \tag{6}$$

5. Similarly, we obtain $$y = \frac{af - ce}{ad - bc}. \tag{7}$$

By the definition of a determinant of the second order, it is verified that this solution, in (6) and (7), can be written as follows:

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}. \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}. \tag{8}$$

Hence, in a system of the form of (1) and (2), where $(ad - bc) \neq 0$, **the value of each unknown is a quotient of two determinants**, where

1. *the denominator determinant is the same for all unknowns, and is formed by writing down the coefficients of the unknowns, in order, and*

2. *the numerator, for any unknown, can be obtained from the denominator by replacing the column of coefficients of this unknown by the constant terms.*

Thus, in (8), the constant terms, e and f, are in the 1st column in the numerator for x, and in the 2d column of the numerator for y.

Example 2. — Solve by determinants: $\begin{cases} 2x - 4y = -14, \\ 3x + 7y = 5. \end{cases}$

Solution. — From (8), $x = \dfrac{\begin{vmatrix} -14 & -4 \\ 5 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & -4 \\ 3 & 7 \end{vmatrix}}$, and $y = \dfrac{\begin{vmatrix} 2 & -14 \\ 3 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & -4 \\ 3 & 7 \end{vmatrix}}$.

Computing the determinants, we obtain

$$x = \frac{-78}{26} = -3; \quad y = \frac{52}{26} = 2.$$

NOTE. — The expressions in (8) are not applicable to a system in which the denominator $(ad - bc) = 0$. It is proved in advanced mathematics that, in case $(ad - bc) = 0$, the equations of the system are either dependent or inconsistent.

EXERCISE 27

Problems 1 to 18. — Solve the problems 1 to 18 of Exercise 23 by use of determinants.

Solve for x and y by use of determinants:

19. $\begin{cases} ax + by = 1, \\ -bx + ay = 1. \end{cases}$

20. $\begin{cases} 5x - 30 = 0, \\ 2x - 3y - 18 = 0. \end{cases}$

21. $\begin{cases} cx + 2dy - 4cd = 0, \\ 2dx + cy = c^2 + 4d^2. \end{cases}$

22. $\begin{cases} 2x = 3y, \\ 5x = 8y + 1. \end{cases}$

23. $\begin{cases} acx - by = 0, \\ abcx + aby = (a+b)^2. \end{cases}$

24. $\begin{cases} bx - ay = 2b^2, \\ x + 2y = 4b + a. \end{cases}$

25. Why cannot formulas 8 be used to solve the adjoining system? Determine graphically whether or not this system has a solution.

$\begin{cases} 3x + 5y = 2, \\ 6x + 10y = 5. \end{cases}$

51. Determinants of the third order. — The symbol

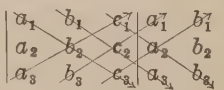
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{1}$$

is called a **determinant of the third order** and is defined as an abbreviation for the expression

$$a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3. \tag{2}$$

The numbers a_1, a_2, a_3, b_1, c_1, etc. are called the **elements**, and (2) is called the **expansion** of the determinant.

NOTE 1. — The terms in (2) can be remembered as follows: Repeat the first two columns at the right of the determinant, as in the diagram. *Multiply together each set of three elements*

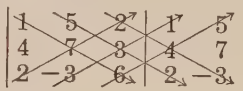


through which a line is drawn. Give the product a plus sign if the line of the product runs down, from left to right: give the product a minus sign if its line runs up, from left to right. The six terms thus obtained are those in (2).

Thus, $\begin{vmatrix} 1 & 5 & 2 \\ 4 & 7 & 3 \\ 2 & -3 & 6 \end{vmatrix} = 1\cdot 7\cdot 6 + 5\cdot 3\cdot 2 + 2\cdot 4\cdot(-3) - 2\cdot 7\cdot 2 - (-3)\cdot 3\cdot 1 - 6\cdot 4\cdot 5$

$= 42 + 30 - 24 - 28 + 9 - 120 = -91.$

The terms in this expansion were obtained by use of the adjoining diagram, which was constructed as directed in Note 1.



EXERCISE 28

Obtain the value of each determinant:

1. $\begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 4 \\ -2 & 1 & -3 \end{vmatrix}$ **2.** $\begin{vmatrix} -2 & 0 & 2 \\ -1 & 3 & 1 \\ 4 & -2 & 3 \end{vmatrix}$ **3.** $\begin{vmatrix} 5 & 3 & 1 \\ 2 & 0 & -2 \\ -1 & 3 & 4 \end{vmatrix}$

Write the expansions of the following determinants:

4. $\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ 2 & 3 & 4 \end{vmatrix}$ **5.** $\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}$ **6.** $\begin{vmatrix} x & 2 & 3 \\ c & y & 1 \\ -1 & 3 & 2 \end{vmatrix}$

52. Solution of three linear equations by determinants.

Example. — Solve for x, y, and z:

$$\begin{cases} a_1x + b_1y + c_1z = d_1, & (1) \\ a_2x + b_2y + c_2z = d_2, & (2) \\ a_3x + b_3y + c_3z = d_3. & (3) \end{cases}$$

Solution. — 1. Multiply (1) by b_2:

$$a_1b_2x + b_1b_2y + b_2c_1z = b_2d_1. \quad (4)$$

2. Multiply (2) by b_1: $a_2b_1x + b_1b_2y + b_1c_2z = b_1d_2.$ (5)

3. (4) − (5): $(a_1b_2 - a_2b_1)x + (b_2c_1 - b_1c_2)z = (b_2d_1 - b_1d_2).$ (6)

4. Eliminating y similarly between (1) and (3), we obtain

$$(a_1b_3 - a_3b_1)x + (b_3c_1 - b_1c_3)z = (b_3d_1 - b_1d_3). \quad (7)$$

5. On eliminating z between (6) and (7) by the method of addition and subtraction, we obtain

$$x = \frac{d_1b_2c_3 + d_3b_1c_2 + d_2b_3c_1 - d_3b_2c_1 - d_1b_3c_2 - d_2b_1c_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}, \quad (8)$$

provided that the denominator is not zero.

6. Similarly we obtain the following values for y and z:

$$y = \frac{a_1d_2c_3 + a_3d_1c_2 + a_2d_3c_1 - a_3d_2c_1 - a_1d_3c_2 - a_2d_1c_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}. \tag{9}$$

$$z = \frac{a_1b_2d_3 + a_3b_1d_2 + a_2b_3d_1 - a_3b_2d_1 - a_1b_3d_2 - a_2b_1d_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}. \tag{10}$$

7. The denominator, for x, y, and z, is identical with the expansion of the determinant 1 of Section 51. The numerator for x differs from the denominator only in that the a's in the denominator are replaced by d's. Hence, the numerator is a determinant which can be obtained from the determinant in the denominator by replacing the a's by d's. Similar remarks apply to the numerators for y and z. Hence,

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}. \tag{11}$$

Note 1. — Steps 1 and 2 in the description of the solution of two linear equations by determinants, in Section 50, apply without a change in wording to the solution of three linear equations by determinants.

The formulas 11 may not be used in solving a system for which the denominator is zero; in advanced mathematics, it is proved that the equations of such a system are either inconsistent or dependent.

Example. — Solve by determinants: $\begin{cases} 3x + y - z = 14, \\ x + 3y - z = 16, \\ x + y - 3z = -10. \end{cases}$

Solution. —

$$x = \frac{\begin{vmatrix} 14 & 1 & -1 \\ 16 & 3 & -1 \\ -10 & 1 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 1 & 1 & -3 \end{vmatrix}} = \frac{-126 + 10 - 16 - 30 + 48 + 14}{-27 - 1 - 1 + 3 + 3 + 3} = \frac{-100}{-20} = 5.$$

$$y = \frac{\begin{vmatrix} 3 & 14 & -1 \\ 1 & 16 & -1 \\ 1 & -10 & -3 \end{vmatrix}}{-20} = \frac{-120}{-20} = 6. \qquad z = \frac{\begin{vmatrix} 3 & 1 & 14 \\ 1 & 3 & 16 \\ 1 & 1 & -10 \end{vmatrix}}{-20} = \frac{-140}{-20} = 7.$$

Historical Note. — Determinants were introduced into mathematics to simplify and systematize the solution of simultaneous linear equations. Determinants were first invented by Leibnitz (1646–1716). His invention did not attract notice, and determinants were reinvented in 1750 by Cramer (1704–1752), whose work forms the beginning of the actual development of the theory of determinants.

EXERCISE 29

Problems 1 to 6. — By use of determinants, solve Problems 1 to 6, Exercise 25.

MISCELLANEOUS EXERCISE 30

Solve by introducing two or more unknowns:

1. A first number plus twice a second number equals 4; the first minus the second equals 8. Find the numbers.

2. The sum of two numbers is 12; the first minus the second equals 8. Find the numbers.

3. When a first number is divided by a second, the quotient is 4 and the remainder is 1. The first plus twice the second equals 10. Find the numbers.

Hint. — See formulas 1 and 2 of Section 9.

4. If 1 is added to both numerator and denominator of a certain fraction, the new fraction equals $\frac{3}{5}$. If 3 is subtracted from both numerator and denominator of the original fraction, the new fraction equals -1. Find the original fraction.

5. The length of a certain rectangle exceeds its width by 7 feet. The perimeter is 54 feet. Find the length and width.

6. A contractor employs 20 workmen, some of whom receive \$5.25 and some \$7.50 per day. If the daily payroll is \$123, how many get each wage?

7. The sum of the digits of a certain two-digit number is 9. If the number * is divided by its units' digit, the result is 36. Find the number.

HINT. — If u is the units' digit and t is the tens' digit, the number equals $(10\,t + u)$. Thus, $38 = 10(3) + 8$.

8. A consignment of 36 packages of paper weighs 125 pounds. There are two kinds of packages, one weighing $2\frac{1}{2}$ pounds and the other $4\frac{1}{4}$ pounds per package. How many packages of each variety are there?

9. If a certain rectangular lot were 6 feet longer and 5 feet wider, the area would be 839 square feet greater, and if the lot were 7 feet wider and 4 feet longer, the area would be 879 square feet greater. Find the present dimensions.

10. The sum of the digits of a certain two-digit number is 6 and if the number is divided by its tens' digit, the result is 12. Find the number.

11. The sum of the digits of a certain two-digit number is 6. If the digits are reversed, the new number obtained is 18 less than the original one. Find the original number.

12. Two angles are complementary, and one exceeds the other by 15. Find the angles.

13. A crew can row $10\frac{1}{2}$ miles per hour downstream and $5\frac{1}{2}$ miles per hour upstream. Find the rate of the current and the rate at which the crew can row in still water. †

14. It takes a motor boat $1\frac{1}{3}$ hours to go 20 miles downstream and $2\frac{2}{3}$ hours to return. Find the rate of the current and the rate of the boat in still water.

15. The sum of the three digits of a certain three-digit number is 16. The sum of the tens' and the units' digits is three times the hundreds' digit. If the digits of the number are reversed, the new number exceeds the old one by 99. Find the original number.

* In digit problems, the word number refers to a positive integer.

† Review Note 1, page 43.

16. It takes a certain man 3 hours to row 20 miles downstream in a river and 10 hours to return. Find the rate of the current and the rate at which the man can row in still water.

17. One sum of money is invested at the rate 4% and another at 7%. The total of the sums invested is \$4000 and the total annual interest is \$211. How much is invested at each rate?

18. The reciprocal of a first number minus the reciprocal of a second number equals 1. The sum of the reciprocals of the numbers is 7. Find the numbers.

19. When working together, A and B can paint a certain house in 8 days. At one time, they painted this house when A worked 12 days and B worked 6 days. How long would it take each to paint the house alone?

20. When working together, A and B can plow a certain field in 6 days. If A worked alone for 5 days, and if B then worked alone for 2 days, only $\frac{2}{3}$ of the field would be plowed. How many days would it take each to plow the field alone?

21. A man has a first sum invested at 4%, a second at 5%, and a third at 6%. The total amount invested is \$10,000 and the total annual interest received is \$490. Find the sum invested at each rate if the amount invested at 5% equals the sum of the amounts invested at the other rates.

CHAPTER VII

EXPONENTS

53. Laws of positive integral exponents. — In Section 5 we defined a^m, when m is a positive integer, by the equation

$$a^m = a \cdot a \cdot a \cdot \cdots \cdot a. \qquad (m \text{ factors})$$

The determination of powers of numbers is called **involution**. Certain laws govern the use of positive integral exponents.

I. Law for multiplication: $a^m a^n = a^{m+n}$.

Illustrations. $a^3 \cdot a^8 = a^{11}$; $a^3 \cdot a^{5x} = a^{3+5x}$.
Review the proof of Law I in Section 6.

II. Law for division:

1. $\dfrac{a^m}{a^n} = a^{m-n}$. (If m is greater than n)

2. $\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$. (If n is greater than m)

Illustrations. $\dfrac{a^8}{a^3} = a^5$; $\dfrac{a^7}{a^{10}} = \dfrac{1}{a^3}$; $\dfrac{a^b}{a^{3b}} = \dfrac{1}{a^{2b}}$.

Review the proof of Part I, of Law II, in Section 8. A similar proof of Part 2 will be asked for in Problem 70, Exercise 31.

III. Law for finding a power of a power: $(a^m)^n = a^{mn}$.

Illustrations. $(a^3)^4 = a^{12}$; $(a^5)^x = a^{5x}$.

Proof. — **1**: $(a^m)^n = a^m \cdot a^m \cdot a^m \cdot \cdots \cdot a^m$ (n factors a^m)

2. Use Law I: $= a^{m+m+m \cdots +m}$ (n terms "m")

3. Since $(m + m + m + \cdots + m)$, to n terms, equals mn, therefore $(a^m)^n = a^{mn}$.

IV. Law for finding a power of a product: $(ab)^n = a^n b^n$.

Illustrations. $(ab)^4 = a^4b^4$; $(a^2d)^3 = (a^2)^3d^3 = a^6d^3$.

Proof. — **1.** $(ab)^n = (ab) \cdot (ab) \cdot (ab) \cdot \cdots \cdot (ab)$. ($n$ factors ab)

2. Rearrange a's and b's: $= (a \cdot a \cdot a \cdot \cdots \cdot a)(b \cdot b \cdot b \cdot \cdots \cdot b)$.
(n factors) (n factors)

3. Hence, by the definition of an exponent, $(ab)^n = a^n b^n$.

NOTE. — Law IV extends to products of any number of factors. Thus, $(abc)^n = a^n b^n c^n$; $(abcd)^n = a^n b^n c^n d^n$; etc.

V. Law for finding a power of a quotient: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

Illustrations. $\left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4}$; $\left(\frac{a^2}{3\,b}\right)^3 = \frac{(a^2)^3}{(3\,b)^3} = \frac{a^6}{27\,b^3}$.

Proof. — **1.** $\left(\frac{a}{b}\right)^n = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdot \cdots \cdot \frac{a}{b}$ $\left(n \text{ factors } \frac{a}{b}\right)$

2. $= \frac{a \cdot a \cdot a \cdot \cdots \cdot a}{b \cdot b \cdot b \cdot \cdots \cdot b}$. ($n$ factors) / (n factors)

3. Hence, by the definition of an exponent, $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

Laws I to V are called the index laws.

EXERCISE 31

1. By use of the definition of an exponent, find the value of 2^6; $(-3)^5$; $(-2)^5$; $(-4)^4$; $(-1)^6$; $(-5)^4$; $(-1)^3$; $(.1)^5$.

Thus, $2^6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 =$ etc.

2. What is the sign of an odd power of a negative number; of an even power of a negative number?

3. Find the value of $(\frac{1}{2})^4$; $(\frac{1}{3})^5$; $(\frac{1}{5})^4$; $(\frac{1}{6})^3$; $(\frac{1}{4})^3$.

Perform the indicated operations, using laws I *to* V:

4. $x^4 \cdot x^{10}$.	**8.** $2^2 \cdot 2^3$.	**12.** $d \cdot d^4 \cdot d^2$.	**16.** $(a^2)^2$.
5. $a^3 \cdot a^7$.	**9.** $x^3 \cdot x^m$.	**13.** $x^3 \cdot x^{n-3}$.	**17.** $(2\,a)^5$.
6. $a^4 \cdot a^b$.	**10.** $a^n \cdot a^{1-n}$.	**14.** $(x^3)^4$.	**18.** $(3\,m)^3$.
7. $c^n \cdot c^h$.	**11.** $c \cdot c^3 \cdot c^4$.	**15.** $(y^3)^5$.	**19.** $(a^4)^n$.

20. a^4a^n.
21. $(-.2x)^3$.
22. $(5^2)^3$.
23. $(ax)^5$.
24. $(a^2b)^2$.
25. $(xy^3)^3$.
26. $(.2a^3)^5$.
27. $(-.7x)^3$.
28. $(-3a)^3$.
29. $(4xy)^3$.
30. $(3x^2yz)^2$.
31. $(-xy^2)^3$.
32. $(-5z^3)^4$.
33. $(.1ab)^5$.
34. $(5x^3)^2$.
35. $5(x^3)^2$.
36. $(\frac{1}{2}x)^2$.
37. $(r^3s)^n$.
38. $(y^m)^{4n}$.
39. $(x^ty^3)^a$.
40. $(.2xy^{3n})^4$.
41. $(-4x^2y)^4$.
42. $a^5(ab^2)^3$.
43. $x^3(x^2y)^4$.

44. $\dfrac{x^{12}}{x^5}$.
45. $\dfrac{y^6}{y^8}$.
46. $\dfrac{z^{10}}{z^3}$.
47. $\dfrac{x^{n+3}}{x^2}$.
48. $\dfrac{a^6}{a^{10}}$.
49. $\dfrac{z^3}{z^{n+3}}$.
50. $\dfrac{y^5z^7}{y^2z^3}$.
51. $\dfrac{a^{16}x^{13}}{ax^2}$.
52. $\dfrac{z^{12}y^5}{z^{14}y^2}$.
53. $\dfrac{w^{2r+1}}{w^2}$.

54. $\left(\dfrac{x}{y}\right)^5$.
55. $\left(\dfrac{x^2}{b}\right)^3$.
56. $\left(\dfrac{2a}{3b}\right)^2$.
57. $\left(\dfrac{-x}{3y}\right)^3$.
58. $\left(\dfrac{x^2}{5y^2}\right)^2$.
59. $\left(-\dfrac{4x}{yz}\right)^3$.
60. $\left(-\dfrac{2c}{d}\right)^7$.
61. $\left(\dfrac{.2x}{.3b}\right)^3$.
62. $\left(\dfrac{4c^2}{x}\right)^3$.
63. $\left(\dfrac{c^2d}{x}\right)^4$.
64. $\left(\dfrac{ab^2}{5c}\right)^3$.
65. $\left(\dfrac{-.12x}{ab}\right)^2$.
66. $\left(\dfrac{x^2}{y^3}\right)^n$.
67. $\left(\dfrac{2x^n}{y^4}\right)^3$.
68. $\left(\dfrac{c^m}{2d^n}\right)^4$.
69. $\left(\dfrac{a^{2n}}{b^{3x}}\right)^2$.

70. Prove Part 2 of Law II of Section 53.

54. Square roots. — We call R a **square root** of A if $R^2 = A$.

Thus, -2 is a square root of 4 because $(-2)^2 = 4$.

If A is positive, A has **two square roots,** one positive and one negative, whose absolute values are equal. The positive square root is called the **principal** square root of A and is denoted by the symbol $\sqrt{A}$.

Thus, the square roots of 9 are 3 and -3, but the symbol $\sqrt{9}$ represents *only the positive root,* 3; $-3 = -\sqrt{9}$.

55. Real and imaginary numbers. — A negative number A has no positive or negative square root because the square of a positive, or of a negative number is positive and could not equal a negative number A. Hence, we agree that, when A is

negative, $\sqrt{A}$ represents a new kind of number, which we shall call an **imaginary** number, with the property that $(\sqrt{A})^2 = A$.

Thus, $\sqrt{-5}$ and $\sqrt{-1}$ are imaginary numbers.

In contrast to imaginary numbers, in the future we shall call positive and negative numbers **real** numbers. *Unless otherwise stated, all literal numbers in this book will represent real numbers.*

56. Roots of any order. — We call R a cube root of A if $R^3 = A$. If n is any positive integer, we say that

R is an nth root of A if $R^n = A$.

Thus, -3 is a cube root of -27 because $(-3)^3 = -27$.
$2a^2$ is a 5th root of $32a^{10}$ because $(2a^2)^5 = 32a^{10}$.

The determination of roots is spoken of as the **extraction** of roots, and is called the process of **evolution**.

Every number A has n distinct nth roots,* some or all of which may be imaginary numbers.

If n is even, every positive number A has two real nth roots, one positive and one negative, whose absolute values are equal. The positive root is called the **principal nth root** of A and is denoted by the symbol $\sqrt[n]{A}$.

Thus, ± 3 are the real 4th roots of 81; $+3$ is the principal 4th root.

If n is odd, every real number A has one and only one real nth root, which is positive when A is positive, and is negative when A is negative. This single real root is called the **principal nth root** of A and is denoted by $\sqrt[n]{A}$.

Thus, the principal cube root of -8 is -2; of 125 is 5.

If n is even, and A is negative, all nth roots of A are imaginary. In this case, when convenient, we use $\sqrt[n]{A}$ to represent one of these imaginary roots.

* The proof of this statement will be given in a later chapter.

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According to the preceding agreements,

I. $\sqrt[n]{A}$ *is positive if* n *is even and* A *is positive;*

II. $\sqrt[n]{A}$ *is positive if* n *is odd and* A *is positive;*

III. $\sqrt[n]{A}$ *is negative if* n *is odd and* A *is negative;*

IV. $\sqrt[n]{A}$ *is imaginary if* n *is even and* A *is negative.*

Thus, $\sqrt[6]{64} = 2$; $\sqrt[5]{32} = 2$; $\sqrt[5]{-32} = -2$; $\sqrt[4]{-64}$ is imaginary.

Unless otherwise specified, **the word root will refer to the principal root**, and "$\sqrt[n]{A}$" will be read "*the* nth root of A."

57. Radicals. — The symbol $\sqrt[n]{A}$ is called a **radical**. That is, a radical is a root of an expression, called the **radicand**, indicated by a radical sign. The integer telling the root represented is called the **index** or the **order** of the radical.

Thus, $\sqrt[4]{16\,x^2}$ is a radical of the 4th order; $16\,x^2$ is the radicand.

NOTE. — In this book, if the index of a radical is an even integer, all literal numbers not used as exponents represent positive numbers.

Notice that, since $(b)^n = b^n$,

$$\sqrt[n]{b^n} = b. \tag{1}$$

Thus, $\sqrt[5]{-32} = \sqrt[5]{(-2)^5} = -2$; $\sqrt[3]{343} = \sqrt[3]{7^3} = 7$.

By the definition of $\sqrt[n]{A}$ as an nth root it is seen that

$$(\sqrt[n]{A})^n = A. \tag{2}$$

Thus, $(\sqrt[3]{5})^3 = 5$; $(\sqrt[4]{x})^4 = x$; $(\sqrt{A^3})^2 = A^3$.

EXERCISE 32

By inspection tell the values of the specified roots:

1. The two square roots of 144; 121; 196; $\frac{1}{4}$; .01; 100; 1.
2. The principal square root of 16; 64; 169.
3. The cube root of -27; 64; -125; 1000; $-.001$; .008.
4. The two real 4th roots of 16; 81; 10,000; 625; $\frac{1}{16}$.
5. The 5th root of 32; -32; 243; -243; 100,000.

Find the value of each radical:

6. $\sqrt{49}$. 9. $\sqrt[4]{16}$. 12. $\sqrt[8]{256}$. 15. $\sqrt{.0004}$.
7. $\sqrt[3]{216}$. 10. $\sqrt[5]{-32}$. 13. $\sqrt[5]{-243}$. 16. $\sqrt[3]{.008}$.
8. $\sqrt[3]{-27}$. 11. $\sqrt[7]{128}$. 14. $\sqrt[3]{-1000}$. 17. $\sqrt{\frac{1}{4}}$.
18. $\sqrt[3]{-\frac{1}{8}}$. 19. $\sqrt[3]{27000}$. 20. $\sqrt[4]{\frac{1}{16}}$.

What is the value of each indicated power:

21. $(\sqrt{7})^2$. 22. $(\sqrt[3]{9})^3$. 23. $(\sqrt[3]{-6})^3$. 24. $(\sqrt{3\,ab})^2$.

58. Fractional exponents. — We have previously defined a^m only when m is a positive integer. We shall now define a^m when m is fractional or negative or zero, in such a way that the new types of powers will obey the laws of positive integral powers.

If a fractional power is to obey Law I of Section 53, then, for example, $a^{\frac{5}{2}}$ should be defined so that $a^{\frac{5}{2}}a^{\frac{5}{2}} = a^5$, or $(a^{\frac{5}{2}})^2 = a^5$, or so that $a^{\frac{5}{2}}$ is a *square root of* a^5. This suggests the following definition.

Definition. — *If m and n are positive integers, then*

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}. \tag{1}$$

Illustrations. $a^{\frac{3}{3}} = \sqrt[3]{a^3}$; $4^{\frac{3}{2}} = \sqrt{4^3} = \sqrt{64} = 8$.

If $m = 1$ in equation 1, we obtain

$$a^{\frac{1}{n}} = \sqrt[n]{a}. \tag{2}$$

Illustration. $\sqrt[5]{3} = 3^{\frac{1}{5}}$; $\sqrt[3]{3\,x^2yz} = (3\,x^2yz)^{\frac{1}{3}}$.

By our definition, every fractional power can be written as a radical and vice versa. Each form is more convenient than the other for certain kinds of manipulation.

59. Zero as an exponent. — If operations with a^0 are to obey Law I of Section 53, then we must define a^0 so that $a^0a^n = a^{0+n}$, or $a^0a^n = a^n$, or $a^0 = a^n \div a^n$. Hence we are led to the following definition.

Definition. — *If a is not zero, then* $\boldsymbol{a^0 = 1}$.

Illustrations. $x^0 = 1$; $4^0 = 1$; $(xyz)^0 = 1$.

60. Negative exponents. — If negative powers are to obey Law I of Section 53, then, if n is a positive number, a^{-n} must be defined so that $a^{-n}a^n = a^{-n+n} = a^0$, or so that $a^{-n}a^n = 1$. This suggests the following definition.

Definition. — $$\boldsymbol{a^{-n} = \frac{1}{a^n}}. \qquad (1)$$

Illustrations. $x^{-5} = \frac{1}{x^5}$; $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$.

61. Negative powers in fractions. — Since $a^{-n}a^n = a^0 = 1$,

$$\frac{c}{xa^{-n}} = \frac{c}{xa^{-n}} \cdot \frac{a^n}{a^n} = \frac{ca^n}{xa^0}; \quad \text{or} \quad \frac{c}{xa^{-n}} = \frac{ca^n}{x}$$

$$\frac{ba^{-m}}{x} = \frac{ba^{-m}}{x} \cdot \frac{a^m}{a^m} = \frac{ba^0}{xa^m}; \quad \text{or} \quad \frac{ba^{-m}}{x} = \frac{b}{xa^m}.$$

Therefore, in a fraction, *any factor of one term* (numerator or denominator) *may be omitted if the factor, with the sign of its exponent changed, is written as a factor of the other term of the fraction.*

Illustrations. $\frac{x^{-4}y^2}{x^3y^{-3}} = \frac{y^2 \cdot y^3}{x^3 \cdot x^4} = \frac{y^5}{x^7}$; $\frac{3\,x^2y}{cd^3} = 3\,x^2yc^{-1}d^{-3}$.

62. Laws of exponents. — In the proof of laws I to V of Section 53, it was understood that all exponents were integers. In advanced mathematics it is proved that these five laws hold when we allow the exponents to have any values of the types introduced in this chapter. *In the future* **use laws I to V in all operations involving exponents.**

EXERCISE 33

Find the value of each expression, by changing each fractional power to a radical, and each negative power to a positive one:

1. $4^{\frac{1}{2}}$. **6.** $(-4)^{-4}$. **11.** $(-1)^{\frac{1}{5}}$. **16.** $(.0144)^{\frac{1}{2}}$.

2. $16^{\frac{1}{4}}$. **7.** $(-8)^{\frac{1}{3}}$. **12.** $(-3)^{-2}$. **17.** $(.00001)^{\frac{1}{5}}$.

3. 5^{-1}. **8.** $(-32)^{\frac{1}{5}}$. **13.** $16^{-\frac{1}{2}}$. **18.** $(1.5)^{-3}$.

4. 4^{-2}. **9.** $(\frac{1}{4})^{\frac{1}{2}}$. **14.** $81^{-\frac{1}{4}}$. **19.** $(.001)^{-\frac{1}{3}}$.

5. $(-5)^{-3}$. **10.** $(\frac{1}{16})^{\frac{1}{4}}$. **15.** $8^{-\frac{1}{3}}$. **20.** $(.0016)^{-\frac{1}{4}}$.

Express with positive exponents:

21. a^{-2}. **23.** $a^{-5}x^2z^{-3}$. **25.** $5\,a^{-3}$. **27.** $5^{-2}x^3$.

22. cd^{-3}. **24.** $a^{-2}x^{-4}$. **26.** $6\,x^{-2}b^{-3}$. **28.** $x^{-2}ax^{-2}$.

29. $\dfrac{5\,x^{-3}y}{z^3}$. **31.** $\dfrac{5\,x^{-4}}{3\,y^{-5}}$. **33.** $\dfrac{10^{-4}x^{-2}}{7}$. **35.** $\dfrac{5}{x^{-4}y^2z^{-2}}$.

30. $\dfrac{a^2b^{-4}}{c^{-4}b^2}$. **32.** $\dfrac{6\,x^2y^{-4}}{2\,x^{-3}y^5}$. **34.** $3\,x^{-4}y^{-4}y^2$. **36.** $\dfrac{5\,x^{-4}y}{3\,z^{-4}x}$.

Write without a denominator, by use of negative powers:

37. $\dfrac{5\,a^6}{y^3}$. **39.** $\dfrac{16\,x^3y^2}{z^3h^2}$. **41.** $\dfrac{1}{ab}$. **43.** $\dfrac{8\,x^{\frac{1}{2}}}{y^{\frac{1}{2}}z}$.

38. $\dfrac{3\,c^4}{a^2}$. **40.** $\dfrac{5}{yz^4}$. **42.** $\dfrac{1}{(1.05)^7}$. **44.** $\dfrac{5\,a^3}{(1.03)^4}$.

Rewrite, expressing each fractional power as a radical:

45. $a^{\frac{3}{4}}$. **47.** $5\,a^{\frac{1}{2}}$. **49.** $(3\,xy)^{\frac{1}{2}}$. **51.** $(2\,ab)^{\frac{2}{3}}$.

46. $x^{\frac{5}{3}}$. **48.** $3\,b^{\frac{2}{3}}$. **50.** $(4\,cd^2)^{\frac{1}{3}}$. **52.** $(-4\,cd)^{\frac{2}{5}}$.

Express each radical as a fractional power:

53. $\sqrt[3]{a^2}$. **55.** $\sqrt[5]{x^3}$. **57.** $\sqrt[7]{A}$. **59.** $\sqrt[4]{a+b}$. **61.** $\sqrt[3]{(a+x)^2}$.

54. $\sqrt{x^7}$. **56.** $\sqrt[4]{z^9}$. **58.** $\sqrt[9]{K}$. **60.** $\sqrt[3]{3\,c+d}$. **62.** $\sqrt{(c-d)^3}$.

NOTE. — **To simplify** an expression involving exponents will mean to perform indicated operations by use of laws I to V of Section 53, and

to express the final result in a form without any zero or negative exponents. Do not introduce radicals unless it is requested.

Simplify and, if no letters are involved, find its value:

63. $x^{\frac{1}{3}}x^2$.

64. $x^{\frac{1}{2}}x^{\frac{1}{4}}$.

65. a^0a^{-3}.

66. $(a^{\frac{1}{3}})^4$.

67. $(x^2y^4)^{\frac{1}{2}}$.

68. $(2^{\frac{3}{2}})^4$.

69. $(5^3)^{\frac{2}{3}}$.

70. $(a^3)^{\frac{1}{3}}$.

71. $(a^{-2}b^4)^{\frac{1}{2}}$.

72. $(a^{-3}b^{-6})^{-1}$.

73. $(2^{\frac{1}{4}}a^b)^4$.

74. $(2^3x^3)^{\frac{2}{3}}$.

75. $(5^3x^{-3})^{\frac{1}{3}}$.

76. $(5^4x^{-s})^{\frac{1}{4}}$.

77. $(5^{-3}x^3)^{\frac{2}{3}}$.

78. $(x^{-2}y^{\frac{3}{4}})^{-\frac{1}{2}}$.

79. $(2^{-3}x^6)^{-\frac{1}{3}}$.

80. $(x^3y^4)^n$.

81. $(x^{2n}a^{3n})^{\frac{1}{n}}$.

82. $(2\,x^{\frac{1}{3}})^n$.

83. $\dfrac{x^5}{x^{\frac{3}{2}}}$.

84. $\dfrac{a}{a^{\frac{7}{2}}}$.

85. $\dfrac{y^0a^3}{a^{\frac{1}{2}}}$.

86. $\dfrac{a^3x^{\frac{1}{2}}}{x^{\frac{1}{4}}}$.

87. $\dfrac{ax}{a^{\frac{1}{3}}x^{\frac{1}{4}}}$.

88. $\dfrac{2\,a^3x^b}{ax^{\frac{5}{2}}}$.

89. $\dfrac{6\,a^2x^{\frac{3}{2}}}{2\,a^3x}$.

90. $\dfrac{6\,a^{\frac{2}{3}}x^0}{a^{-\frac{1}{4}}}$.

91. $\dfrac{8\,x^{\frac{1}{2}}}{2\,x^{\frac{1}{3}}y^{-2}}$.

92. $\left(\dfrac{x^2}{y^4}\right)^{\frac{1}{2}}$.

93. $\left(\dfrac{2\,x^2}{y^{\frac{1}{2}}}\right)^2$.

94. $\left(\dfrac{3\,x^{\frac{1}{3}}}{2\,y^{\frac{2}{3}}}\right)^3$.

Hint. — In simplifying, it is advisable to write each integral number in the expression as a power or product of powers of prime integers.

Thus, $\left(\dfrac{216\,x^2}{125\,x^{-1}y^6}\right)^{\frac{2}{3}} = \left(\dfrac{8\cdot 27\cdot x^2\cdot x}{5^3y^6}\right)^{\frac{2}{3}} = \left(\dfrac{2^3\cdot 3^3x^3}{5^3y^6}\right)^{\frac{2}{3}}$

$= \dfrac{2^23^2x^2}{5^2y^4} = \dfrac{36\,x^2}{25\,y^4}$.

95. $(125)^{\frac{2}{3}}$.

96. $(27)^{\frac{2}{3}}$.

97. $(216)^{\frac{4}{3}}$.

98. $(64)^{\frac{3}{2}}$.

99. $\left(\dfrac{27\,m^3}{z^6}\right)^{\frac{2}{3}}$.

100. $\left(\dfrac{16\,x^4y^5}{z^4x^2}\right)^{\frac{1}{2}}$.

101. $\left(\dfrac{m^{6x}}{121\,x^{\frac{4}{3}}}\right)^{\frac{1}{2}}$.

102. $\left(\dfrac{3\,x^{\frac{2}{3}}y^3}{2\,x^{\frac{5}{3}}y^{-8}}\right)^3$.

103. $\left(\dfrac{3\,a^{\frac{1}{2}}b^{\frac{1}{3}}}{16\,c^2}\right)^{\frac{1}{2}}$.

104. $\left(\dfrac{.001\,z^3y^{-2}}{y^4}\right)^{\frac{1}{3}}$.

105. $\left(\dfrac{243\,a^5}{32\,x^{-10}}\right)^{\frac{1}{5}}$.

106. $\left(\dfrac{3^6a^{\frac{7}{2}}x^0}{y^{-6}}\right)^{\frac{2}{3}}$.

107. $\left(\dfrac{.125\,x^{-3}}{8\,z^6}\right)^{\frac{2}{3}}$.

Illustration. $\dfrac{a^{-2}+b^{-2}}{a^{-1}+b^{-1}} = \dfrac{\dfrac{1}{a^2}+\dfrac{1}{b^2}}{\dfrac{1}{a}+\dfrac{1}{b}} = \dfrac{\dfrac{b^2+a^2}{a^2b^2}}{\dfrac{b+a}{ab}} = \text{etc.}$

108. $\dfrac{5}{x^{-1}+y^{-1}}$. **110.** $\dfrac{x^{-4}y^{-4}}{x^{-2}+y^{-2}}$. **112.** $\dfrac{a^{-1}-b^{-1}}{a^{-1}b^{-1}}$.

109. $\dfrac{6x}{x^{-2}+y^{-2}}$. **111.** $\dfrac{1-5^{-3}}{6}$. **113.** $\dfrac{y^{-2}-(\frac{1}{2})^{-2}}{y^{-2}}$.

Obtain indicated products or powers, and simplify the result:

114. $2xy(x^{-2}-3x^{-1}y^{-1}+y^{-2})$. **119.** $(e^x-e^{-x})^2$.

115. $3x^{\frac{5}{2}}(x^{\frac{1}{2}}-5x^{\frac{3}{2}}-4x^{\frac{5}{2}})$. **120.** $(x^{\frac{1}{2}}+y^{\frac{1}{2}})(x^{\frac{1}{2}}+3y^{\frac{1}{2}})$.

116. $(x^{-1}+2x^{-3})(x^{-2}+5x^{-4})$. **121.** $(x^{-1}+y^{-1})^2$.

117. $(a^{\frac{4}{5}}-a^{\frac{3}{5}}x+a^{\frac{2}{5}}x^2)(a^{\frac{1}{5}}+x)$. **122.** $(x^{\frac{1}{2}}-2y^{\frac{1}{3}})^2$.

118. $(x^{-1}-xy)^2$. **123.** $(x^{\frac{1}{3}}-y^{\frac{1}{3}})(x^{\frac{1}{3}}+y^{\frac{1}{3}})$.

124. $(x^{\frac{1}{2}}+y^{\frac{1}{2}})(x^{\frac{3}{2}}-4xy^{\frac{1}{2}}+4x^{\frac{1}{2}}y)$.

125. $(a^{\frac{1}{3}}-3a^{\frac{1}{3}}b^{\frac{1}{3}})(a^{\frac{2}{3}}-b^{\frac{2}{3}})$. **128.** $(x^{-1}y^{-1}+x^2y^{\frac{3}{2}})^2$.

126. $(x^{\frac{1}{2}}+y^{\frac{1}{2}})(x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y^{\frac{1}{2}})$. **129.** $(3a^{\frac{1}{2}}-2a^{\frac{1}{3}})(3a^{\frac{1}{2}}+2a^{\frac{1}{3}})$.

127. $(x^{\frac{1}{3}}+y^{\frac{1}{3}})(x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}})$. **130.** $(c^2-d^{\frac{1}{4}})(c^2+d^{\frac{1}{4}})$.

131. $(x^{\frac{1}{3}}+x^{\frac{1}{6}}+1)(x^{\frac{1}{3}}-1+x^{\frac{1}{6}})$.

132. $(x^{-1}+3y^{-1}-xy^{-2})(x^{-1}-2y^{-1}+xy^{-2})$.

133. $(3x^{-\frac{1}{3}}-1)(9x^{-\frac{2}{3}}+3x^{-\frac{1}{3}}+1)$.

SUPPLEMENTARY PROBLEMS

Factor into two factors involving fractional or negative powers:

Illustration. $x-3x^{\frac{1}{2}}y^{\frac{1}{2}}-4y=(x^{\frac{1}{2}}-4y^{\frac{1}{2}})(x^{\frac{1}{2}}+y^{\frac{1}{2}})$.

134. x^2-y^{-2}. **136.** $4x^{-4}-b^{-4}$. **138.** $16x^{\frac{2}{3}}-64y^{\frac{2}{3}}$.

135. $x^{-2}-y^{-2}$. **137.** $y^4-9x^{\frac{8}{3}}$. **139.** $x^{\frac{2}{5}}-9y^{\frac{2}{5}}$.

140. $9x^{-2} - 6x^{-1} + 1$.

141. $x^{\frac{1}{2}} + 2x^{\frac{1}{4}} + 1$.

142. $4a^{\frac{2}{3}} + 12a^{\frac{1}{3}}b^{\frac{1}{3}} + 9b^{\frac{2}{3}}$.

143. $a^2 + 2ab^{-1} + b^{-2}$.

144. $a^{-2} - 5a^{-1} - 6$.

145. $3 - 4x^{\frac{1}{3}} + x^{\frac{2}{3}}$.

Find the indicated quotients by use of factoring:

146. $\dfrac{x^{-2} - y^{-2}}{x^{-1} - y^{-1}}$.

147. $\dfrac{x^{-4} - 4y^{-2}}{x^{-2} + 2y^{-1}}$.

148. $\dfrac{9x - y}{3x^{\frac{1}{2}} - y^{\frac{1}{2}}}$.

149. $\dfrac{a^{2x} - c^{2y}}{a^x + c^y}$.

150. $\dfrac{a - b}{a^{\frac{1}{3}} - b^{\frac{1}{3}}}$.

151. $\dfrac{a^{\frac{3}{2}} - b^{\frac{3}{2}}}{a^{\frac{1}{2}} - b^{\frac{1}{2}}}$.

Perform the indicated divisions, obtaining a result not involving fractions:

152. $\dfrac{6x^{\frac{1}{2}}y - 15x}{3x^{\frac{1}{2}}}$.

153. $\dfrac{3x^{\frac{1}{3}}y - 4x^{\frac{2}{3}}y^{\frac{2}{3}}}{x^{\frac{1}{3}}y^{\frac{1}{3}}}$.

154. $\dfrac{4r^{\frac{3}{2}} - 6rs - 12s^{\frac{3}{2}}}{2r^{\frac{1}{2}}s^{\frac{1}{2}}}$.

155. $\dfrac{6c^2 - 9cd - 15d^2}{3cd^{-1}}$.

156. $(x^{-3} + 4x^{-2}y - x^{-1}y^2 - 4y^3) \div (x^{-1} + y)$.

HINT. — Use the long division process of Section 9.

157. $(x^{\frac{3}{2}} - 4x + x^{\frac{1}{2}} + 2) \div (x^{\frac{1}{2}} - 1)$.

158. $(x^{\frac{3}{4}} + 4x^{\frac{1}{2}}y^{\frac{1}{2}} + 4x^{\frac{1}{4}}y + y^{\frac{3}{2}}) \div (x^{\frac{1}{4}} + y^{\frac{1}{2}})$.

159. $(m^4 - 2m^2 + 1 + 4m^{-2}) \div (m^{-2} + 1)$.

160. $(2y^{-3} - 9y^{-2} + 17y^{-1} - 14) \div (y^{-1} - 2)$.

161. $(5y^{-3} - 12y^{-1} + 3y^{-2} + 4) \div (5y^{-2} - 7y^{-1} + 2)$.

162. $(12a^{\frac{3}{5}} - 80a^{\frac{2}{5}} + 175a^{\frac{1}{5}} - 125) \div (2a^{\frac{1}{5}} - 5)$.

NOTE. — By Property III, of Section 53, $a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m$, or

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m \qquad (1)$$

That is, $a^{\frac{m}{n}}$, which we defined as the nth root of a^m, also equals the mth *power of* $\sqrt[n]{a}$. Formula 1 is useful in evaluating expressions.

Thus, $(-125)^{\frac{2}{3}} = (\sqrt[3]{-125})^2 = (-5)^2 = 25$.

By use of the preceding note, find the value of each power:

163. $9^{\frac{3}{2}}$. **165.** $16^{\frac{3}{4}}$. **167.** $(.001)^{\frac{2}{3}}$. **169.** $(-\frac{1}{125})^{\frac{4}{3}}$.

164. $8^{\frac{4}{3}}$. **166.** $25^{\frac{5}{2}}$. **168.** $(-128)^{\frac{3}{7}}$. **170.** $(\frac{1}{16})^{\frac{5}{4}}$.

CHAPTER VIII

RADICALS

63. Rational and irrational numbers. — *A real number which can be expressed as the quotient of two integers,* $\frac{a}{b}$, is called a **rational** number. If a is any integer, then $a = \frac{a}{1}$. Hence, *all positive and negative integers and zero are included among the rational numbers.* A real number which cannot be expressed as the quotient of two integers is called an **irrational** number.

Thus, 4, -5, $\frac{16}{3}$, 0, and $-\frac{61}{99}$ are rational numbers; π, $\sqrt{2}$, and $\sqrt[3]{5}$ are irrational numbers. $\sqrt[n]{a}$ is irrational if a is not the nth power of a rational number. In advanced mathematics it is proved that every irrational number can be expressed as an unending, but not a repeating decimal. In numerical computation with irrational numbers we use approximations to their values. Thus, an approximate value of $\sqrt{3}$ is 1.732, correct to three decimal places.

A **surd** is an irrational number which is a root of a rational number. That is, if a is rational and $\sqrt[n]{a}$ is irrational, we call $\sqrt[n]{a}$ a surd of the nth order. A surd of the second order is called a **quadratic** surd.

Thus, $\sqrt{3}$ is a quadratic surd; $\sqrt[6]{64}$ is not a surd because $\sqrt[6]{64} = 2$.

64. Rational functions. — A **rational function** of certain variables is a function which can be expressed as the *quotient of two integral rational polynomials in the variables.*

Thus, $\frac{a^2 - 3b^2}{2ac}$ is a rational function of a, b, and c.

A rational function is called a **perfect *n*th power** if the function is the nth power of some other rational function. A number is called a perfect nth power if it is the nth power of some rational number.

Thus, $243\,x^5y^{10}$ is a perfect 5th power because $243\,x^5y^{10} = (3\,xy^2)^5$.

65. Properties of radicals.

Property I. — $(\sqrt[n]{a})^n = a$. (Def. of an nth root)

Property II. — $\sqrt[n]{a^m} = a^{\frac{m}{n}}$. (Def. of $a^{\frac{m}{n}}$)

Illustration. $\sqrt[4]{x^{12}} = x^{\frac{12}{4}} = x^3$.

Property III. — $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$.

Illustration. $\sqrt[3]{8\,a^6} = \sqrt[3]{8}\sqrt[3]{a^6} = 2\,a^2$.

Proof. — $\sqrt[n]{ab} = (ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}} = \sqrt[n]{a}\sqrt[n]{b}$.

Property IV. — $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

Illustrations. $\sqrt{\frac{9}{25}} = \frac{\sqrt{9}}{\sqrt{25}} = \frac{3}{5}$; $\sqrt{\frac{4\,x^2}{9}} = \frac{\sqrt{4\,x^2}}{\sqrt{9}} = \frac{2\,x}{3}$.

Proof. — $\sqrt[n]{\frac{a}{b}} = \left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

When the radicand is a perfect nth power, $\sqrt[n]{A}$ can be obtained by use of Properties II, III, and IV.

For convenience, express all integral numbers in a radicand as powers or as products of powers of integers.

Illustration. $\sqrt{\frac{625\,x^8}{y^4z^{12}}} = \sqrt[4]{\frac{5^4x^8}{y^4z^{12}}} = \frac{\sqrt[4]{5^4x^8}}{\sqrt[4]{y^4z^{12}}}$ (Prop. IV)

$= \frac{\sqrt[4]{5^4}\sqrt[4]{x^8}}{\sqrt[4]{y^4}\sqrt[4]{z^{12}}} = \frac{5\,x^2}{yz^3}$. (Prop. III; II)

66. Method for removing factors from a radicand:

1. *Separate the radicand into factors as many of which as possible are perfect* nth *powers, where* n *is the index of the radical.*

2. *Find the indicated root of each perfect power and use Property III of Section 65 to express the final result.*

Illustration 1. $\sqrt{147} = \sqrt{49 \cdot 3} = \sqrt{49}\sqrt{3} = 7\sqrt{3}$.

From Table I, $\sqrt{3} = 1.732$; $\sqrt{147} = 7(1.732) = 12.124$.

Illustration 2. $3\sqrt[5]{-729\,a^{10}c^9} = -3\sqrt[5]{243\,a^{10}c^5 \cdot 3\,c^4}$

$= -3\sqrt[5]{3^5 a^{10} c^5}\sqrt[5]{3\,c^4} = -3 \cdot 3\,a^2 c\sqrt[5]{3\,c^4} = -9\,a^2 c\sqrt[5]{3\,c^4}$.

In the future, simplify all radicals by removing all possible perfect powers from the radicands.

EXERCISE 34

Find the indicated roots:

1. $\sqrt{36 \cdot 64}$.
2. $\sqrt{\frac{49}{121}}$.
3. $\sqrt[3]{-1}$.
4. $\sqrt[4]{\frac{625}{16}}$.
5. $\sqrt[4]{16 \cdot 625}$.
6. $\sqrt[3]{-\frac{1}{27}}$.
7. $\sqrt[4]{\frac{1}{16}}$.
8. $\sqrt{\frac{225}{81}}$.
9. $\sqrt[3]{-\frac{1}{64}}$.
10. $\sqrt{\frac{1}{225}}$.

Remove perfect powers from the radicands and then compute by use of Table I:

11. $\sqrt{175}$.
12. $3\sqrt{150}$.
13. $\sqrt{108}$.
14. $\sqrt[3]{24}$.
15. $5\sqrt{200}$.
16. $\sqrt[3]{-135}$.
17. $2\sqrt[3]{-320}$.
18. $\sqrt[3]{250}$.
19. $3\sqrt[3]{432}$.
20. $4\sqrt{.98}$.
21. $\sqrt{450}$.
22. $\sqrt{72}$.

Find the indicated roots by use of Properties II to IV:

23. $\sqrt[4]{16}$.
24. $\sqrt[3]{y^3 z^9}$.
25. $\sqrt[3]{-27\,z^6}$.
26. $\sqrt[5]{-\frac{x^5}{32}}$.
27. $\sqrt[5]{\frac{z^{10}w^5}{32}}$.
28. $\sqrt[4]{256\,z^8}$.
29. $\sqrt[3]{-8\,w^9}$.
30. $\sqrt[4]{16\,a^4 b^8}$.
31. $\sqrt[3]{\frac{a^3}{8}}$.
32. $\sqrt[3]{\frac{r^6}{27}}$.
33. $\sqrt[3]{8\,y^{3k}}$.
34. $\sqrt[4]{x^{4m}y^8}$.
35. $\sqrt[5]{-x^5 y^{10}}$.
36. $\sqrt[4]{\frac{16\,a^8}{x^4 y^4}}$.
37. $\sqrt[4]{\frac{a^8 y^4}{16\,y^{12}}}$.
38. $\sqrt[5]{32\,y^{5k}}$.
39. $\sqrt[6]{a^{12n}y^{6n}}$.
40. $\sqrt[n]{x^{2n}y^{3n}}$.
41. $\sqrt[5]{\frac{-1}{243\,x^5}}$.
42. $\sqrt[3]{\frac{125}{-8\,a^6}}$.

Remove all perfect powers from the radicands:

43. $6\sqrt[4]{32\,m^4}$. **47.** $\sqrt[8]{256\,x^{16}y^{8n}}$. **51.** $\sqrt[5]{243\,a^2z^3}$. **55.** $\sqrt[7]{-128\,x^{14}}$.

44. $3\sqrt[3]{108\,a^7}$. **48.** $\sqrt[5]{32\,x^5y^6}$. **52.** $10\sqrt{.04\,x^3}$. **56.** $8\sqrt{\tfrac{1}{4}\,x^7y^8}$.

45. $\sqrt[3]{-128\,t^6}$. **49.** $4\sqrt[3]{375\,m^7}$. **53.** $\sqrt[3]{.008\,a^4}$. **57.** $\sqrt{.80\,c^4d^3}$.

46. $5\sqrt[4]{80\,c^3d^5}$. **50.** $3\sqrt{.49\,m^4z}$. **54.** $\sqrt{.36\,x^7}$. **58.** $\sqrt[3]{-27\,x^{3k}}$.

59. $\sqrt{4(1+x)}$. **61.** $\sqrt{a^2b-a^2c}$. **63.** $\sqrt[4]{d^8+3\,c^2d^4}$.

60. $\sqrt{9+9\,x^2}$. **62.** $\sqrt{c^2d^2+5\,d^2}$. **64.** $\sqrt[3]{16\,a^3+8\,a^2z^2}$.

65. $\sqrt{\dfrac{4\,x^3}{y^4}}$. **67.** $\sqrt{\dfrac{5\,x^{2k}}{4\,y^4}}$. **69.** $\sqrt[5]{\dfrac{-32\,x^{5n}}{243}}$. **71.** $\sqrt[4]{\dfrac{32\,x^{4n}}{y^{4k}}}$.

66. $\sqrt[3]{\dfrac{3\,x^{3m}}{y^{3n}}}$. **68.** $\sqrt[3]{\dfrac{6\,x^5}{64\,y^9}}$. **70.** $\sqrt{\dfrac{125\,x^3}{9\,y^6}}$. **72.** $\sqrt[n]{\dfrac{1}{y^{mn}}}$.

In the radical $\sqrt{b^2-4\,ac}$, *substitute each of the following sets of values and remove perfect squares from the radicands:*

73. $a=3,\ b=8,\ c=3$. **76.** $a=-10,\ b=s,\ c=5$.

74. $a=-5,\ b=6,\ c=9$. **77.** $a=4\,m,\ b=4\,m,\ c=2$.

75. $a=-2,\ b=4,\ c=-1$. **78.** $a=-2,\ b=2\,k,\ c=k$.

67. Rationalizing the denominator in a radical. — A fraction whose denominator is an irrational number is inconvenient to compute, or to manipulate algebraically.

Thus, $\sqrt{\dfrac{3}{7}}=\dfrac{\sqrt{3}}{\sqrt{7}}$ is an inconvenient form.

To rationalize the denominator in a radical of the *n*th order, *multiply both numerator and denominator of the radicand by the most simple expression that will make the denominator a perfect* nth *power.*

Illustration 1. $\sqrt{\dfrac{3}{7}}=\sqrt{\dfrac{3\cdot 7}{7^2}}=\dfrac{\sqrt{21}}{7}=\dfrac{4.583}{7}=.655.$

Illustration 2. $\sqrt[3]{\dfrac{64}{9\,x^4}}=\sqrt[3]{\dfrac{4^3\cdot 3\,x^2}{3^2x^4\cdot 3\,x^2}}=\sqrt[3]{\dfrac{4^3\cdot 3\,x^2}{3^3x^6}}=\dfrac{4\sqrt[3]{3\,x^2}}{3\,x^2}.$

EXERCISE 35

Express each decimal number as a fraction; rationalize all denominators and then compute by use of Table I:

1. $\sqrt{\frac{1}{2}}$.
2. $\sqrt{\frac{3}{2}}$.
3. $2\sqrt{\frac{1}{3}}$.
4. $\sqrt{\frac{5}{3}}$.
5. $\frac{1}{3}\sqrt{\frac{5}{6}}$.
6. $3\sqrt{\frac{5}{11}}$.
7. $\sqrt{\frac{1}{7}}$.
8. $5\sqrt{\frac{3}{10}}$.
9. $8\sqrt[3]{\frac{1}{4}}$.
10. $\sqrt[3]{\frac{2}{5}}$.
11. $\sqrt[3]{\frac{3}{1000}}$.
12. $\sqrt{\frac{7}{1000}}$.
13. $\sqrt{.3}$.
14. $\sqrt{.05}$.
15. $\sqrt[3]{.128}$.
16. $\sqrt[3]{.06}$.
17. $\sqrt{.7}$.
18. $\sqrt[3]{.24}$.

Rationalize the denominators:

19. $\sqrt{\frac{5}{7x}}$.
20. $5\sqrt{\frac{3a}{m}}$.
21. $\sqrt{\frac{xy^2}{20}}$.
22. $\sqrt{\frac{5a^3}{2x^5}}$.
23. $\sqrt{\frac{5x}{6b}}$.
24. $\sqrt{\frac{5}{8x^2}}$.
25. $\sqrt[4]{\frac{cx}{d}}$.
26. $\sqrt[3]{\frac{1}{4z^2}}$.
27. $3\sqrt[3]{\frac{5a}{3t^3}}$.
28. $\sqrt[3]{\frac{8s^3}{3w^4}}$.
29. $\sqrt[3]{-\frac{5}{4d}}$.
30. $\sqrt[6]{\frac{a^2y^8}{16b^5}}$.
31. $\sqrt{\frac{1}{5x^2y}}$.
32. $h\sqrt[3]{\frac{2}{3hz}}$.
33. $w^2\sqrt[3]{\frac{1}{7uw}}$.
34. $\sqrt[5]{\frac{-1}{81z}}$.
35. $\sqrt{\frac{2-3z^2}{6z}}$.
36. $\sqrt{x^2+\frac{3}{2x}}$.
37. $\sqrt{\frac{3\overline{ab}}{c-d}}$.
38. $\sqrt[4]{\frac{8+5y}{8x^3}}$.

68. The order of a radical sometimes can be reduced, as follows:

1. *Express the radicand as a power of some rational function.*

2. *Express the radical as a fractional power of this function, simplify the exponent, and change the result into a radical.*

Illustration 1. $\sqrt[9]{64} = \sqrt[9]{2^6} = (2^6)^{\frac{1}{9}} = 2^{\frac{2}{3}} = \sqrt[3]{2^2} = \sqrt[3]{4}$.

Illustration 2. $\sqrt[12]{64x^6} = \sqrt[12]{(2x)^6} = [(2x)^6]^{\frac{1}{12}} = (2x)^{\frac{1}{2}} = \sqrt{2x}$.

69. Method for reducing a radical to its simplest form:

1. *Reduce the radicand to a simple fraction in its lowest terms, involving only positive powers.*

2. *Rationalize the denominator.*

3. *Remove from the radicand all factors which are perfect* nth *powers, where* n *is the index of the radical.*

4. *Reduce the final radical to the lowest possible order.*

Illustration. $\sqrt{5\,x^{-3}+3\,a}=\sqrt{\dfrac{5}{x^3}+3\,a}$

$$=\sqrt{\frac{5+3\,ax^3}{x^3}}=\sqrt{\frac{(5+3\,ax^3)x}{x^4}}=\frac{\sqrt{5\,x+3\,ax^4}}{x^2}.$$

EXERCISE 36

Reduce to a lower order, and compute by use of Table I:

1. $\sqrt[6]{25}$. **2.** $\sqrt[4]{36}$. **3.** $\sqrt[10]{32}$. **4.** $\sqrt[6]{343}$. **5.** $\sqrt[4]{49}$. **6.** $\sqrt[6]{16}$.

Reduce to a radical of lower order:

7. $\sqrt[4]{25\,x^2}$. **10.** $\sqrt[4]{9\,a^2}$. **13.** $\sqrt[6]{.001\,x^3}$. **16.** $\sqrt[4]{49\,z^2}$.

8. $\sqrt[10]{32\,a^5}$. **11.** $\sqrt[6]{27\,a^3b^3}$. **14.** $\sqrt[9]{8\,z^3}$. **17.** $\sqrt[6]{.008\,y^3}$.

9. $\sqrt[8]{16\,m^2}$. **12.** $\sqrt[4]{100\,a^2}$. **15.** $\sqrt[8]{16\,x^4y^4}$. **18.** $\sqrt[4]{225\,x^2}$.

19. $\sqrt[12]{196\,x^8y^8}$. **20.** $\sqrt[4]{.0144\,a^2}$. **21.** $\sqrt[12]{64\,a^6b^6}$.

Reduce to simplest form:

22. $\sqrt[4]{4\,y^6}$.

23. $\sqrt[6]{8\,a^3y^9}$.

24. $3\sqrt{\frac{2}{3}}$.

25. $\sqrt[3]{\frac{2}{5}}$.

26. $\sqrt[4]{\frac{1}{9}\,x^2}$.

27. $7\sqrt[6]{16\,x^2}$.

28. $2\sqrt[4]{\frac{4}{25}}$.

29. $5\sqrt{3\,x^{-2}}$.

30. $\sqrt[3]{\frac{1}{5}\,x^7}$.

31. $\sqrt{\dfrac{x^2y^3}{x-y}}$.

32. $\sqrt{\dfrac{1}{x^2}+16}$.

33. $\sqrt{4+\dfrac{4\,a}{x^2}}$.

34. $\sqrt{4\,x^{-3}z^4}$.

35. $\sqrt{\dfrac{1}{a}+b^{-1}}$.

36. $\sqrt{5-\dfrac{c^2}{d^2}}$.

37. $\sqrt{\dfrac{a+2}{a-2}}$.

38. $10\sqrt{\dfrac{3-x}{25\,b}}$.

39. $\sqrt[6]{\dfrac{(a+b)^3}{125\,c}}$.

40. $6\sqrt[3]{-\frac{5}{9}+x^{-2}}$.

41. $\sqrt{\dfrac{1}{a^2}-\dfrac{1}{9\,b^2}}$.

42. $\sqrt{a^2x^{-2}+2\,ax^{-1}+1}$.

43. $\sqrt{\dfrac{c^2}{x^2}-8\,cx^{-1}+16}$.

70. Addition and subtraction of radicals. — We say that two radicals are the same when they have the **same radicand** and the **same index.**

Thus, $\sqrt{5}$ and $\sqrt[3]{5}$ are not the same radical.

In an algebraic sum, we may collect all terms involving the *same* radical into one term.

Illustrations. $3\sqrt{2} - 7\sqrt{2} + 6\sqrt{2} = (3 - 7 + 6)\sqrt{2} = 2\sqrt{2}.$

$a\sqrt{2} + b\sqrt{2} + b\sqrt{3} = (a + b)\sqrt{2} + b\sqrt{3}.$

The addition or subtraction of terms involving *different* radicals can only be indicated.

Thus, $(5\sqrt{3} - 3\sqrt{2})$ cannot be simplified algebraically.

To add or subtract radical expressions, proceed as follows:

1. *Reduce each radical to its simplest form.*
2. *Combine, and collect terms involving the same radical.*

Illustration. $\sqrt[3]{\frac{1}{4}} - \sqrt[3]{16} + \sqrt[3]{81} = \sqrt[3]{\frac{2}{8}} - \sqrt[3]{8 \cdot 2} + \sqrt[3]{27 \cdot 3}$

$= \frac{1}{2}\sqrt[3]{2} - 2\sqrt[3]{2} + 3\sqrt[3]{3} = -\frac{3}{2}\sqrt[3]{2} + 3\sqrt[3]{3}$

$= -\frac{3}{2}(1.260) + 3(1.442) = -1.890 + 4.326 = 2.436$

EXERCISE 37

Simplify, and then compute by use of Table I:

1. $5\sqrt{2} + 2\sqrt{2}.$
2. $3\sqrt{3} - 2\sqrt{3}.$
3. $\sqrt{18} + \sqrt{8}.$
4. $\sqrt{32} + \sqrt{12}.$
5. $\sqrt{147} + \sqrt{\frac{1}{3}}.$
6. $\sqrt{\frac{1}{2}} + \sqrt{50}.$
7. $\frac{1}{6}\sqrt{\frac{5}{3}} - 10\sqrt{\frac{1}{5}} + \sqrt{45}.$
8. $10\sqrt{\frac{2}{5}} - \sqrt{10} + 3\sqrt{8}.$
9. $2.5\sqrt{6} - \sqrt{\frac{2}{3}} + 2\sqrt{2}.$

Simplify and collect terms:

10. $\sqrt{75} - \sqrt{147} + \sqrt{\frac{49}{3}}.$
11. $3\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{3}} + \sqrt{8}.$
12. $5\sqrt{\frac{1}{7}} - \sqrt{63} + \sqrt{28}.$
13. $\sqrt[3]{24} + 2\sqrt[3]{81} - 5\sqrt[3]{3}.$
14. $3\sqrt{a} - \sqrt{25a} + \sqrt{4a}.$
15. $2\sqrt{18x} - \sqrt{50x} + \sqrt{x^2b}.$

16. $a\sqrt{2} + b\sqrt{2} + 3\sqrt{2}$.

17. $3\sqrt{x} - 2a\sqrt{x} - b\sqrt{x}$.

18. $3a\sqrt{16a} - \sqrt{144a^3}$.

19. $\sqrt[5]{729} - \sqrt[5]{3}$.

20. $\sqrt[3]{xy} - 3\sqrt[3]{27a^3b^3xy}$.

21. $\sqrt[4]{81x^5} + \sqrt[9]{y^8x^{10}}$.

22. $3m\sqrt{49m^3n} - 2n\sqrt{64n^3m}$.

23. $5x\sqrt[3]{\frac{1}{4}x} + 5\sqrt[3]{2x^4}$.

24. $\sqrt[4]{48x^2y^4z} - \sqrt{5x^2y} - \sqrt[4]{243x^2z^5} + x\sqrt{20y}$.

25. $2\sqrt{\frac{5}{x}} + 4\sqrt[3]{\frac{27}{16x^2}} - \sqrt{\frac{16}{5}x} - \frac{3}{4}\sqrt[3]{\frac{4}{x^2}}$.

26. $\frac{1}{3}\sqrt{\frac{9a}{2}} + \sqrt[3]{\frac{27a}{4}} - 6\sqrt{\frac{a}{18}} + 2\sqrt[3]{-\frac{a}{32}}$.

27. $2y\sqrt{\frac{x}{y}} + xy\sqrt{\frac{x^2 + y^2 + 2xy}{xy}} - 3x\sqrt{\frac{y}{x}}$.

28. $\sqrt{\frac{x-1}{x+1}} - \sqrt{\frac{x+1}{x-1}} - \frac{1}{x^2-1}\sqrt{4x^2 - 4}$.

71. Multiplication of radicals of the same order is accomplished by the formula of Property III, Section 65:

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}. \tag{1}$$

This formula extends to products of three or more radicals.

Thus, $\sqrt[n]{a}\sqrt[n]{b}\sqrt[n]{c} = \sqrt[n]{ab}\sqrt[n]{c} = \sqrt[n]{abc}$.

Always reduce the product radical to its simplest form.

Illustrations. $(2\sqrt{3})(5\sqrt{6}) = 10\sqrt{3}\sqrt{6} = 10\sqrt{18} = 30\sqrt{2}$
$= 30(1.414) = 42.42$

$\sqrt[3]{2a}\sqrt[3]{20a^2b^3} = \sqrt[3]{40a^3b^3} = \sqrt[3]{5 \cdot 8a^3b^3} = 2ab\sqrt[3]{5}$.

Recall that, since the radical $\sqrt[n]{A}$ represents an nth root, $(\sqrt[n]{A})^n = A$. In particular, $(\sqrt{A})^2 = A$; $(\sqrt[3]{A})^3 = A$, etc.

EXERCISE 38

Reduce to simplest form; then compute by use of Table I:

1. $\sqrt{2}\sqrt{10}$.
2. $\sqrt{3}\sqrt{15}$.
3. $\sqrt[3]{5}\sqrt[3]{50}$.
4. $\sqrt{7}\sqrt{14}$.
5. $\sqrt[3]{72}\sqrt[3]{6}$.
6. $\sqrt{7}\sqrt{2}$.
7. $(3\sqrt{2})\sqrt{6}$.
8. $(4\sqrt{34})(3\sqrt{2})$.

Reduce the following products to simplest form:

9. $\sqrt{3}\sqrt{5\,a}$.
10. $\sqrt[3]{5}\sqrt[3]{15\,x}$.
11. $\sqrt{2\,a}\sqrt{10\,ab}$.
12. $(2\sqrt{y})(3\sqrt{5\,y})$.
13. $\sqrt{5}\sqrt{15}\sqrt{6}$.
14. $\sqrt[3]{x}\sqrt[3]{6\,x^2}$.
15. $\sqrt{3}\sqrt{2\,x}\sqrt{2\,x}$.
16. $(\sqrt{5\,x})^2$.
17. $(2\sqrt{6\,a})^2$.
18. $\sqrt{2\,xy}\sqrt{22\,x^3y^2}$.
19. $(2\sqrt[3]{3})(3\sqrt[3]{5})$.
20. $\sqrt[3]{2\,a}\sqrt[3]{2\,a}\sqrt[3]{2\,a}$.
21. $(5\sqrt[3]{2\,x})^3$.
22. $(5\sqrt[3]{x^2})^3$.
23. $(\sqrt{x^2+y})^2$.
24. $\sqrt[5]{3\,x^3yz^2}\sqrt[5]{x^2y^4z^3}$.
25. $\sqrt[4]{8\,x^2y^2}\sqrt[4]{6\,x^3y^3}$.
26. $(3\sqrt{x+1})^2$.
27. $(-\sqrt{3\,x})(-\sqrt{5\,x})$.
28. $(-2\sqrt[3]{31\,a^2})^3$.
29. $(.5\sqrt{6})(.2\sqrt{3})$.
30. $.6\sqrt[3]{3\,a^2b^2}\sqrt[3]{18\,a^2b^2}$.
31. $\sqrt{3\,x}\sqrt{5\,x}\sqrt{15\,x}$.
32. $\sqrt[3]{x^2}\sqrt[3]{xy}\sqrt[3]{3\,x}$.
33. $(2\sqrt{3}+3\sqrt{2\,x})(3\sqrt{3}-\sqrt{2\,x})$.

First Solution of Problem 33. —

$$\begin{array}{r} 2\sqrt{3}+3\sqrt{2\,x} \\ 3\sqrt{3}-\sqrt{2\,x} \\ \hline 18+9\sqrt{6\,x} \\ \quad -2\sqrt{6\,x}-6\,x \\ \hline 18+7\sqrt{6\,x}-6\,x \end{array}$$

Second Solution. — By use of Special Product II, Section 10,

$$(2\sqrt{3}+3\sqrt{2\,x})(3\sqrt{3}-\sqrt{2\,x}) = 6(\sqrt{3})^2+(9-2)\sqrt{3}\sqrt{2\,x}-3(\sqrt{2\,x})^2$$
$$= 18+7\sqrt{6\,x}-6\,x.$$

34. $(\sqrt{3}+2\sqrt{2})(\sqrt{3}-5\sqrt{2})$.
35. $(2\sqrt{3}-\sqrt{5})(\sqrt{3}+\sqrt{5})$.
36. $(3-2\sqrt{2})(3-2\sqrt{2})$.
37. $(\sqrt{5}-4)^2$.
38. $(1-\sqrt{2})(3+\sqrt{2})$.
39. $(2\sqrt{5}-3)(6-\sqrt{7})$.
40. $(\sqrt{5}-\sqrt{2\,x})^2$.
41. $(\sqrt{3}+2\,y)^2$.
42. $(\sqrt{a+1}-2)^2$.
43. $(2\sqrt{5}-\sqrt{3})(2\sqrt{5}+\sqrt{3})$.

44. $(\sqrt{5x} - 2)(\sqrt{5x} + 2)$.

45. $(6 - \sqrt{2a})(6 + \sqrt{2a})$.

46. $(\sqrt{x} - \sqrt{y})(2\sqrt{x} - 3\sqrt{y})$.

47. $(2\sqrt{3} - \sqrt{ax})(2\sqrt{3} + \sqrt{ax})$.

48. $5(\sqrt{2} - 1)^2$.

49. $(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})$.

50. $(\sqrt{a} - \sqrt{a+1})(\sqrt{a} + \sqrt{a+1})$.

51. $(3\sqrt[3]{x} - 2)(2\sqrt[3]{x} + 2)$.

52. $\sqrt[3]{x}(3\sqrt[3]{x} - 5)$.

53. $(\sqrt{x+2} - \sqrt{x+5})^2$.

54. Find the value of $(x^2 - 6x + 7)$ when $x = (\sqrt{3} + 2)$.

55. Compute the expression $(x^2 - 6x + 7)$ when $x = 3 - \sqrt{2}$.

56. Compute the expression $(ax^2 + bx + c)$ when

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

Replace the coefficient of each radical by an equivalent factor under the radical sign:

Illustration. $5\sqrt{7a} = \sqrt{5^2}\sqrt{7a} = \sqrt{175a}$.

57. $3\sqrt{6h}$.

58. $5\sqrt{\frac{2a^2}{5}}$.

59. $6\sqrt[3]{4hw}$.

60. $5a\sqrt{6b^2}$.

61. $3c\sqrt[3]{4dh}$.

62. $2x\sqrt{\frac{5}{2x}}$.

63. $3\sqrt[4]{5ab}$.

64. $6c\sqrt{\frac{5bc}{3}}$.

72. Division of one radical by another of the same order can be performed by use of Property IV of Section 65:

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}. \tag{1}$$

The resulting radical should be simplified.

Illustration. $\frac{2\sqrt[3]{3ab}}{5\sqrt[3]{b^2}} = \frac{2}{5}\sqrt[3]{\frac{3ab}{b^2}} = \frac{2}{5}\sqrt[3]{\frac{3ab^2}{b^3}} = \frac{2}{5b}\sqrt[3]{3ab^2}$.

73. We agree that an expression involving radicals is in its simplest form if the following facts are true:

1. *Indicated powers and products have been simplified.*
2. *Each radical has been simplified, as in Section 69.*
3. *The denominators of all fractions have been rationalized.*

Example 1. — Simplify: $(3\sqrt{2}-\sqrt{3})\div(2\sqrt{2}-\sqrt{3})$.

Solution. —
$$\frac{3\sqrt{2}-\sqrt{3}}{2\sqrt{2}-\sqrt{3}}=\frac{3\sqrt{2}-\sqrt{3}}{2\sqrt{2}-\sqrt{3}}\cdot\frac{2\sqrt{2}+\sqrt{3}}{2\sqrt{2}+\sqrt{3}}$$
$$=\frac{6(\sqrt{2})^2+(3-2)\sqrt{6}-(\sqrt{3})^2}{(2\sqrt{2})^2-(\sqrt{3})^2}=\frac{12+\sqrt{6}-3}{8-3}$$
$$=\frac{9+\sqrt{6}}{5}=\frac{9+2.449}{5}=2.290.$$

Any quotient which can be simplified by the method of Section 67 can also be simplified by rationalizing the divisor by the method of Example 1.

Illustration. $\dfrac{1}{\sqrt[3]{2}}=\dfrac{1}{\sqrt[3]{2}}\cdot\dfrac{\sqrt[3]{4}}{\sqrt[3]{4}}=\dfrac{\sqrt[3]{4}}{\sqrt[3]{8}}=\dfrac{\sqrt[3]{4}}{2}$.

EXERCISE 39

Simplify by use of equation 1, *Section* 72, *and compute:*

1. $\dfrac{\sqrt{15}}{\sqrt{5}}$.
2. $\dfrac{\sqrt{14}}{\sqrt{2}}$.
3. $\dfrac{3\sqrt{3}}{\sqrt{6}}$.
4. $\dfrac{\sqrt{3}}{\sqrt{7}}$.
5. $\dfrac{3\sqrt[3]{5}}{\sqrt[3]{2}}$.
6. $\dfrac{\sqrt[3]{5}}{\sqrt[3]{\frac{1}{3}}}$.
7. $\dfrac{5\sqrt{5}}{2\sqrt{3}}$.
8. $\dfrac{\sqrt{10\,x}}{\sqrt{2\,x}}$.
9. $\dfrac{2\sqrt[3]{99}}{3\sqrt[3]{11}}$.
10. $\dfrac{10\sqrt[3]{5}}{\sqrt[3]{\frac{2}{5}}}$.

Perform indicated divisions by use of equation 1, *Section* 72:

11. $\dfrac{3\,a\sqrt{10\,b^2}}{\sqrt{5\,ab^5}}$.
12. $\dfrac{\sqrt[4]{48\,x^8}}{\sqrt[4]{12\,x^9}}$.
13. $\dfrac{\sqrt[3]{60\,a^3}}{\sqrt[3]{120}}$.
14. $\dfrac{\sqrt[3]{12\,ay^2}}{y\sqrt[3]{6\,a^3y}}$.
15. $\sqrt{50\,x}\div\sqrt{20\,x}$.
16. $\sqrt[5]{39\,a^2}\div\sqrt[5]{13\,x^6}$.
17. $\sqrt{14\,y^3}\div\sqrt{\frac{1}{7}\,y^2}$.
18. $\sqrt[3]{8\,tz^2}\div\sqrt[3]{32\,t^2z^5}$.
19. $\sqrt{\frac{1}{2}x^4y}\div\sqrt{\frac{1}{4}\,xy^2}$
20. $144\sqrt[3]{x^2y}\div 36\sqrt[3]{\frac{1}{8}\,y}$.
21. $\sqrt[6]{25\,x^2y}\div\sqrt[6]{125\,xy^3}$.
22. $(4\sqrt{12}-3\sqrt{18})\div\sqrt{6}$.
23. $(5\sqrt[3]{4\,x}-3\sqrt[3]{2\,x})\div\sqrt[3]{8\,x}$.
24. $\sqrt[n]{x^2yz}\div\sqrt[n]{x^{n-2}y^{n-3}}$.

Rationalize the denominator and compute by use of Table I:

25. $\frac{2}{\sqrt{5}}$.
26. $\frac{1}{\sqrt{5}}$.
27. $\frac{6}{\sqrt{2}}$.
28. $\frac{\sqrt{3}}{2\sqrt{5}}$.
29. $\frac{3\sqrt{5}}{4\sqrt{7}}$.
30. $\frac{2}{\sqrt{15}}$.
31. $\frac{3\sqrt[3]{4}}{\sqrt[3]{9}}$.
32. $\frac{2}{3\sqrt[3]{25}}$.
33. $\frac{1}{\sqrt{30}}$.
34. $\frac{50}{\sqrt{.1}}$.
35. $\frac{6\sqrt{140}}{\sqrt{.2}}$.
36. $\frac{\sqrt{3\,x}}{\sqrt{5\,x}}$.
37. $\frac{5}{\sqrt{2}-\sqrt{3}}$.
38. $\frac{6}{2\sqrt{3}-5\sqrt{2}}$.
39. $\frac{2\sqrt{2}}{3\sqrt{2}-2}$.
40. $\frac{\sqrt{5}-1}{\sqrt{2}-\sqrt{5}}$.
41. $\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{3}+\sqrt{2}}$.
42. $\frac{\sqrt{5}-1}{2\sqrt{2}-3}$.

Simplify by any convenient method:

43. $\frac{1}{3}\frac{\sqrt{3\,a^3}}{\sqrt{36\,ab}}$.
44. $\frac{\sqrt{2}+3\sqrt{3}}{\sqrt{2}-\sqrt{3}}$.
45. $\frac{2-x}{\sqrt{x-1}-2}$.
46. $\frac{\sqrt{c^2d^3}}{\sqrt{5\,cd}}$.
47. $\sqrt[3]{5} \div \sqrt[3]{\frac{1}{2}}$.
48. $\frac{\sqrt{a}-\sqrt{x}}{\sqrt{a}+\sqrt{x}}$.
49. $\frac{\sqrt{3}+2}{\sqrt{a+b}}$.
50. $\frac{.6}{2\sqrt{.2}-\sqrt{.3}}$.
51. $\frac{6\sqrt{10}-\sqrt{.1}}{2\sqrt{10}-\sqrt{.1}}$.
52. $\sqrt[n]{3\,ax^n} \div \sqrt{2\,ax^{2n}}$.
53. $\sqrt{6\,a} \div (\sqrt{a+b}-\sqrt{b})$.
54. $(2\sqrt{c}-\sqrt{a}) \div (3\sqrt{c}-\sqrt{2\,a})$.
55. $(3+\sqrt[3]{4}) \div 3\sqrt[3]{2}$.
56. $2\sqrt[3]{3\,ab^2} \div \sqrt[3]{25\,a^2b^2}$.
57. $1 \div \sqrt[3]{-2\,a^2b}$.
58. $\sqrt{x-2} \div (\sqrt{3}-\sqrt{x+2})$.
59. $(2\sqrt[3]{2\,xy}-5\sqrt[3]{18\,x^2z}) \div \sqrt[3]{9\,x^2y}$.

74. To change radicals of different orders to equivalent radicals of the same order, proceed as follows:

1. *Express each radical as a fractional power.*
2. *Change the exponents to fractions with their L.C.D.*
3. *Express the powers as radicals whose index is this L.C.D.*

Illustration. $\sqrt{3\,a} = (3\,a)^{\frac{1}{2}} = (3\,a)^{\frac{3}{6}} = \sqrt[6]{(3\,a)^3} = \sqrt[6]{27\,a^3}.$

$\sqrt[3]{5\,a^2} = (5\,a^2)^{\frac{1}{3}} = (5\,a^2)^{\frac{2}{6}} = \sqrt[6]{(5\,a^2)^2} = \sqrt[6]{25\,a^4}.$

75. To find the product or the quotient of two radicals of different orders, first reduce the radicals to equal radicals of the same orders and then multiply, or divide, by the method of Section 71 or of Section 72.

Illustration. $\sqrt[3]{4\,b^2x}\sqrt{3\,ab} = (4\,b^2x)^{\frac{1}{3}}(3\,ab)^{\frac{1}{2}} = (4\,b^2x)^{\frac{2}{6}}(3\,ab)^{\frac{3}{6}}$

$= \sqrt[6]{(4\,b^2x)^2}\sqrt[6]{(3\,ab)^3} = \sqrt[6]{16\,b^4x^2 \cdot 27\,a^3b^3} = b\sqrt[6]{16 \cdot 27\,a^3bx^2}.$

EXERCISE 40

Simplify the following products and quotients:

1. $\sqrt[3]{3}\sqrt{2}.$
2. $\sqrt[4]{3}\sqrt[5]{5}.$
3. $\sqrt{3}\sqrt[3]{7}.$
4. $\sqrt{a}\sqrt[3]{b}.$
5. $\sqrt{3\,x} \div \sqrt[3]{y}.$
6. $\sqrt{6} \div \sqrt[4]{2}.$
7. $\sqrt[4]{7} \div \sqrt{14}.$
8. $\sqrt{5} \div \sqrt[3]{10}.$
9. $2\sqrt[3]{12} \div \sqrt{3}.$
10. $\sqrt[3]{2\,m^2}\sqrt[4]{8\,m^3}.$
11. $\sqrt{2\,y}\sqrt[3]{4\,y^2}.$
12. $\sqrt[3]{5\,m^2z^2}\sqrt{3\,m}.$
13. $\sqrt[3]{x^2y^2} \div 3\sqrt{y}.$
14. $\sqrt[3]{a^2} \div \sqrt{3\,ab}.$
15. $\sqrt[3]{\frac{1}{4}} \div \sqrt{\frac{1}{2}}.$
16. $\sqrt{a - b}\sqrt[4]{a + b}.$
17. $\sqrt{\frac{2}{x}}\sqrt[3]{\frac{3}{x^2}}.$
18. $\sqrt{3}\sqrt[3]{5}\sqrt[4]{2}.$
19. $(\sqrt{2}\sqrt[3]{4\,x}) \div \sqrt[4]{5\,x}.$
20. $\sqrt[3]{a^2 - b^2} \div \sqrt{a + b}.$
21. $\sqrt{\frac{x}{2\,a}}\sqrt[4]{\frac{2\,x}{a}}.$
22. $\sqrt[3]{2\,x} \div (\sqrt{5} - \sqrt{3\,x}).$
23. $\sqrt[4]{3\,a^2} \div (\sqrt{a} - 2\sqrt{b}).$

76. Method for simplifying a power or a root of a radical:

1. *Express each indicated root as a fractional power.*

2. *By use of the laws of exponents, write the given expression as a power of the radicand.*

3. *Change the result of Step 2 to radical form and simplify.*

Illustration 1. $\sqrt[3]{\sqrt[4]{3\,xy}} = [(3\,xy)^{\frac{1}{4}}]^{\frac{1}{3}} = (3\,xy)^{\frac{1}{12}} = \sqrt[12]{3\,xy}.$

Illustration 2. $(2\sqrt{3\,xy^3})^3 = [2(3\,xy^3)^{\frac{1}{2}}]^3 = 2^3(3\,xy^3)^{\frac{3}{2}}$

$= 8\sqrt{(3\,xy^3)^3} = 8 \cdot 3\,xy^3\sqrt{3\,xy^3} = 24\,xy^4\sqrt{3\,xy}.$

EXERCISE 41

Simplify:

1. $(2\sqrt{2})^2$.
2. $(\sqrt[3]{3})^3$.
3. $(\sqrt[3]{9})^2$.
4. $(5\sqrt[3]{18})^3$.
5. $\sqrt{\sqrt{6}}$.
6. $\sqrt[3]{\sqrt{5}}$.
7. $\sqrt{\sqrt[3]{4}}$.
8. $\sqrt{\sqrt[3]{7}}$.
9. $(2\sqrt[3]{b})^4$.
10. $(\sqrt{3\,a})^3$.
11. $\sqrt[3]{\sqrt{ab^2c}}$.
12. $\sqrt{\sqrt[3]{25\,x^2}}$.
13. $\sqrt[3]{a^3\sqrt{8}}$.
14. $\sqrt[4]{16\sqrt[4]{81\,a^4b^2}}$.
15. $\sqrt{4\sqrt[4]{25\,a^8}}$.
16. $\sqrt[3]{8\sqrt{x^3y^3}}$.
17. $\sqrt[3]{\sqrt{\sqrt{3}}}$.
18. $(\sqrt[3]{243\,a^7})^2$.
19. $(3\,a^2\sqrt[3]{m^2})^2$.
20. $(x\sqrt[3]{y^2})^4$.
21. $(2\sqrt[6]{a^5b^3c^2})^2$.
22. $\sqrt[4]{\sqrt[3]{64\,x^5y^3}}$.
23. Prove that $\sqrt[3]{\sqrt{A}} = \sqrt{\sqrt[3]{A}}$.
24. Prove that $\sqrt[m]{\sqrt[n]{A}} = \sqrt[n]{\sqrt[m]{A}}$.

MISCELLANEOUS EXERCISE 42

REVIEW OF CHAPTERS V TO VIII

Find the value of each expression, using Table I if necessary:

1. $(-3)^4$.
2. $(-2)^0$.
3. 5^{-2}.
4. $(1000)^{-1}$.
5. $(.1)^5$.
6. $(64)^{\frac{1}{3}}$.
7. $\sqrt[4]{81}$.
8. $\sqrt[3]{-64}$.
9. $\sqrt[8]{256}$.
10. $(\sqrt{7})^2$.
11. $(\sqrt[3]{137})^3$.
12. $\sqrt{98}$.
13. $\sqrt{\frac{1}{3}}$.
14. $(1000)^{\frac{1}{3}}$.
15. $(144)^{\frac{1}{2}}$.
16. $(81)^{-2}$.
17. $(-8)^{-\frac{2}{3}}$.
18. $(25)^{-\frac{5}{2}}$.
19. $5\sqrt{200}$.
20. $\sqrt{63}$.
21. $\dfrac{1}{\sqrt{5}}$.
22. $\sqrt{\dfrac{1}{27}}$.
23. $(\frac{1}{8})^{\frac{2}{3}}$.
24. $\sqrt[3]{-\frac{1}{64}}$.
25. $\dfrac{3\sqrt{3}}{\sqrt{75}}$.
26. $\sqrt{\frac{5}{3}}$.
27. $5\sqrt{\frac{4}{15}}$.
28. $\sqrt{\frac{4}{15}}$.
29. $\dfrac{1}{\sqrt{3}-\sqrt{2}}$.
30. $\dfrac{2\sqrt{2}}{3\sqrt{2}-2}$.
31. $\dfrac{\sqrt{3}-\sqrt{2}}{2\sqrt{3}+\sqrt{2}}$.
32. $\dfrac{\sqrt{5}-1}{2\sqrt{2}-3}$.
33. $\dfrac{2\sqrt{27}}{\sqrt{5}}$.
34. $\dfrac{\sqrt{5\,a}}{\sqrt{125\,a}}$.
35. $\sqrt[6]{5^3}$.
36. $\sqrt[6]{8}$.

Express without radicals, or negative or zero exponents, and simplify by use of the laws of exponents:

37. $(2\,ab)^4$.

38. $(-3\,x^{\frac{1}{2}}y)^4$.

39. $5^{-1}a^{-1}x^2$.

40. $(x^0y^3)^{5n}$.

41. $(x^2y^4)^{\frac{3}{2}}$.

42. $(2\,x^2y^{\frac{1}{3}})(3\,xy^{\frac{2}{3}})$.

43. $(x^3y^{\frac{4}{3}})^3$.

44. $\sqrt{4\,y}$.

45. $\sqrt[3]{a^2}$.

46. $\sqrt{x^7}$.

47. $16\sqrt{a}$.

48. $\sqrt[3]{ab}$.

49. $(x^2y)^{-2}$.

50. $\sqrt[5]{a^2x^3}$.

51. $\sqrt[3]{27\,x^2yz}$.

52. $\left(\dfrac{27\,a^3}{64\,x^6}\right)^{\frac{1}{3}}$.

53. $\left(\dfrac{5^{-2}w^{-4}}{y^4}\right)^{\frac{1}{2}}$.

54. $\left(\dfrac{5\,z^5x^4}{a^3w^0}\right)^{\frac{1}{2}}$.

55. $\left(\dfrac{c^m}{dx}\right)^n$.

56. $\dfrac{\sqrt{10\,a^{16}x^3}}{\sqrt{5\,ax^2}}$.

57. $\left(\dfrac{3\,x}{4\,y^{\frac{1}{3}}}\right)^3$.

58. $\dfrac{y^{-4}z^{-4}}{y^{-2}+z^{-2}}$.

59. $\dfrac{x^{-1}-y^{-1}}{x^{-2}y^{-2}}$.

60. $\dfrac{5^{-2}-3^{-2}}{3^{-1}+5^{-1}}$.

Reduce to the simplest radical form:

61. $\sqrt[4]{16\,x^4y^8}$.

62. $\sqrt[5]{-x^5w^{10}}$.

63. $\sqrt[3]{-8\,y^3z^5}$.

64. $\sqrt[n]{x^{2n}z^{4n}}$.

65. $\sqrt[3]{\frac{1}{8}\,w^3}$.

66. $\sqrt[3]{-64\,x^9}$.

67. $\sqrt[5]{32\,xy^6}$.

68. $\sqrt[3]{-.008\,z^5}$.

69. $(8\,c^4x^3)^{\frac{1}{3}}$.

70. $\sqrt[5]{64\,a^5}$.

71. $\sqrt{\frac{1}{5}}$.

72. $(\sqrt{\frac{1}{2}})^3$.

73. $\left(\dfrac{16\,w^6}{5\,x^2}\right)^{\frac{1}{2}}$. **74.** $\left(\dfrac{-125}{4\,x^4}\right)^{\frac{1}{3}}$. **75.** $\left(\dfrac{5\,x^{2k}}{3\,y}\right)^{\frac{3}{4}}$. **76.** $\left(\dfrac{5\,x^3}{2\,y^5}\right)^{\frac{2}{3}}$.

77. $3\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}+\sqrt{8}$. **78.** $\sqrt{\frac{1}{7}}+\sqrt{63}-\sqrt{28}$.

79. $\sqrt[4]{48\,y^2xz^4}-\sqrt{5\,xy^2}+\sqrt[4]{243\,y^2x^5}$.

80. $5(3\sqrt{2}-3\sqrt{5})(4\sqrt{2}-\sqrt{10})$.

81. $5\sqrt{\dfrac{3}{25\,b}-1}$. **82.** $\sqrt{4+\dfrac{4\,a}{x^2}}$. **83.** $\sqrt{\dfrac{x+3}{x-2}}$. **84.** $\sqrt[3]{\dfrac{2}{5}}$.

85. $(\sqrt{3\,x})^4$.

86. $\sqrt{5}\sqrt{5\,x}\sqrt{5\,x^3}$.

87. $\sqrt[3]{3\,x^4y^2}\sqrt[3]{3\,xy^4}$.

88. $(2\sqrt{x+1})^2$.

89. $\sqrt[3]{x^2}\sqrt[3]{xy}$.

90. $(\sqrt{x+2}-3)^2$.

91. $\dfrac{\sqrt{2-x}}{\sqrt{x+1}+1}$.

93. $\dfrac{\sqrt{3}+2}{\sqrt{a}+\sqrt{b}}$.

95. $\dfrac{5\sqrt{x}}{3+\sqrt{x}}$.

92. $\dfrac{\sqrt{5}}{3\sqrt{5\,a}}$.

94. $\dfrac{\sqrt[3]{c^2d^3}}{\sqrt[3]{c^4}}$.

96. $\dfrac{1}{\sqrt{30}}$.

97. $\sqrt[5]{\sqrt[3]{a}}$.

98. $\sqrt[4]{16\sqrt[3]{a^2}}$.

Reduce to a radical of lower order:

99. $\sqrt[4]{100\,z^2}$. **100.** $\sqrt[4]{49\,x^4w^2}$. **101.** $\sqrt[6]{125\,x^9}$. **102.** $\sqrt[10]{32\,a^5y^5}$.

103. $\sqrt[4]{4}$. **104.** $\sqrt[6]{8}$. **105.** $\sqrt[4]{9}$. **106.** $\sqrt[8]{16}$. **107.** $\sqrt[6]{27}$. **108.** $\sqrt[4]{25}$.

Simplify, reducing to radicals of the same order when necessary:

109. $\sqrt{a}\sqrt[3]{b}$.

111. $\sqrt[3]{2\,y^2}\sqrt[4]{8\,y^3}$.

113. $\sqrt[3]{4}\sqrt{y^3z^3}$.

110. $\sqrt{3\,x}\sqrt[3]{x}$.

112. $\sqrt[3]{\frac{1}{4}} \div \sqrt{\frac{1}{2}}$.

114. $(a\sqrt[3]{y^2})^4$.

115. $\sqrt[3]{x^2-y^2} \div \sqrt{x+y}$.

116. $\sqrt[3]{2\,a} \div (5-\sqrt{3\,a})$.

117. Draw a graph of $3\,x + 2\,y - 3 = 0$.

118. Draw the locus of the equation $-2\,x + 3\,y = 2$.

Solve algebraically; in **119** *to* **124**, *also solve graphically:*

119. $\begin{cases} 3\,x - y = -12, \\ 4\,x + y = -2. \end{cases}$

120. $\begin{cases} w + 6\,x = -10, \\ 2\,w + 7\,x = -15. \end{cases}$

121. $\begin{cases} x + y = 11, \\ 2\,x + 2\,y = -5. \end{cases}$

122. $\begin{cases} x + 3\,y = 6, \\ 2\,x + 6\,y + 1 = 0. \end{cases}$

123. $\begin{cases} 5\,x - 2\,y = 10, \\ x + y = 9. \end{cases}$

124. $\begin{cases} 5\,x - 3\,y = -9, \\ 3\,x + 4\,y = -17. \end{cases}$

125. $\begin{cases} \dfrac{5}{x-2} - \dfrac{7}{3-2\,y} = 0, \\ \dfrac{1}{2\,x+5} - \dfrac{8}{7-3\,y} = 0. \end{cases}$

126. $\begin{cases} \dfrac{2\,a}{3} - \dfrac{3\,b}{4} = -\dfrac{7}{2}, \\ \dfrac{a}{4} + \dfrac{2\,b}{5} = \dfrac{11}{2}. \end{cases}$

127. $\begin{cases} 3x - y + z = 11, \\ x - 3y + z = 13, \\ x - y + 3z = 11. \end{cases}$ **128.** $\begin{cases} 12x + 4n - z = 3, \\ x + n + 2z = -1, \\ 5x + 2n = 0. \end{cases}$

129. If 1 is subtracted from both numerator and denominator of a certain fraction, the resulting fraction equals $\frac{2}{3}$. If the numerator of the original fraction is multiplied by 2, and if 1 is added to the denominator, the new fraction equals $\frac{5}{4}$. Find the original fraction.

130. A man has \$2.30 in nickels and dimes. If he has 33 coins in all, how many of each sort has he?

131. A man has one sum invested at 3% and another invested at 4%. The total annual interest he receives is \$270. If he had invested both sums at 5%, the annual interest would be \$375. Find the sums invested at each rate.

132. On a river, the rate of whose current is $2\frac{1}{2}$ miles per hour, some men rowed downstream for 2 hours. It took them 6 hours to row back to their starting point. How far did the men row downstream and at what rate could they row in still water?

133. Two automobiles start toward each other from points 120 miles apart and meet in 2 hours. The automobiles then start together in the same direction and, after 2 hours, the faster is 40 miles ahead of the slower one. At what speed does each travel?

134. A and B wish to determine their weight by use of a teeter board. The board balances when A is $5\frac{1}{2}$ feet and B is 7 feet from the point where the board is supported. If A takes a five-pound weight in his pocket, the board balances when A is 6 feet and B is 8 feet from the point of support. Find their weights.

135. The sum of the reciprocals of a certain three numbers is $-\frac{1}{6}$. The reciprocal of the first exceeds that of the second by $\frac{1}{6}$. The reciprocal of the first minus twice the reciprocal of the third is $\frac{5}{2}$. Find the numbers.

136. A and B, working together, did a certain job in 9 days. At another time, they completed the same job when A worked $4\frac{1}{2}$ days and B worked 15 days. How many days would it take each to finish the job, working alone?

CHAPTER IX

ELEMENTS OF QUADRATIC EQUATIONS

77. A quadratic equation is an integral rational equation in which the term of highest degree is of the second degree. In this chapter, each quadratic equation mentioned will involve only a single variable.

Thus, $3x^2 - 5x = 27$ is a quadratic equation in x.

78. A pure quadratic equation in x is one involving no term of the first degree in x.

Thus, $3x^2 - 7 = 0$ is a pure quadratic equation.

Example. — Solve $x^2 = A$.

Solution. — 1. Extract the square roots of both members; the square roots of x^2 are $\pm x$; and those of A are $\pm\sqrt{A}$.

2. $x^2 = A$ if (either $+x$, or $-x$) = (either $+\sqrt{A}$, or $-\sqrt{A}$).

3. The final equation of Step 2 is equivalent to four equations:

$$\text{I.} \begin{cases} x = +\sqrt{A}. \\ x = -\sqrt{A}. \end{cases} \qquad \text{II.} \begin{cases} -x = +\sqrt{A}. \\ -x = -\sqrt{A}. \end{cases}$$

4. If both members of each equation in Set II are multiplied by -1, we obtain $x = -\sqrt{A}$ and $x = +\sqrt{A}$, which are the equations of Set I.

5. Hence, on extracting the square roots of both members of $x^2 = A$, we obtain only two essentially distinct equations, which give the two solutions $x = \pm\sqrt{A}$.

Example. — Solve: $5y^2 = 35 - 2y^2$.

Solution. — 1. Add $2y^2$: $7y^2 = 35$.

2. Divide by 7: $y^2 = 5$.

3. Extract square roots of both sides: $y = \pm\sqrt{5} = \pm 2.236$.

Whenever the square root of a known number occurs in the solution of a quadratic equation, *obtain the value of the root from Table* I *and compute the decimal value of the solution.*

EXERCISE 43

Solve:

1. $3x^2 = 75$. **3.** $7x^2 = 6 - 3x^2$. **5.** $5w^2 - 7 = 0$.

2. $2x^2 - 3 = 0$. **4.** $2x^2 = 69 - x^2$. **6.** $3x^2 - .15 = 0$.

7. $\frac{2x}{3} + \frac{3}{x} = \frac{x}{12} + \frac{12}{x}$. **8.** $\frac{2y - 3}{4 - 3y} = \frac{3 + y}{3y + 2}$.

Solve for x and simplify the resulting radical:

9. $3b - 4ax^2 = 5c$.

Partial Solution. — 1. Subtract $3b$: $-4ax^2 = 5c - 3b$.

2. Divide by $-4a$: $x^2 = \frac{5c - 3b}{-4a} = \frac{3b - 5c}{4a}$.

10. $4x^2 - 3A + B = 0$. **13.** $2x^2 - 5d = cx^2 + b$.

11. $5a + 3bx^2 - 2c = 0$. **14.** Solve $f = \frac{mv^2}{r}$ for v.

12. $ax^2 + 2bx^2 = 3cd$. **15.** Solve $S = \frac{1}{2}gt^2$ for t.

16. Solve $a^2 = b^2 + c^2$ (I) for a; (II) for b; (III) for c.

79. Some quadratic equations can be solved by factoring, as follows:

1. *Simplify and transpose all terms to the left member.*
2. *Factor the left member, if possible.*
3. *Place each factor equal to zero, and solve for the variable.*

Example 1. — Solve $6x^2 + 5x = 6$.

Solution. — 1. $6x^2 + 5x - 6 = 0$.

2. Factor: $(3x - 2)(2x + 3) = 0$.

3. The product of two or more numbers is zero if any one of the numbers is zero. Hence, *since the right member in Step* 2 *is zero,* a value of x is a solution if and only if it makes one or other of the two factors of the left member equal to zero.

4. If $3x - 2 = 0$, $3x = 2$; hence $x = \frac{2}{3}$ is one solution.
5. If $2x + 3 = 0$, $2x = -3$; hence $x = -\frac{3}{2}$ is a second solution.

Example 2. — Solve: $4x^2 + 20x + 25 = 0$.

Solution. — 1. Factor: $(2x + 5)(2x + 5) = 0$, or $(2x + 5)^2 = 0$.
2. When $2x + 5 = 0$, $x = -\frac{5}{2}$.
3. Since the other factor also is $(2x + 5)$, it yields the same value for x, and hence **the given equation has two equal roots.**

NOTE. — In solving an equation involving a single variable, if both members are divided by an expression involving the variable, solutions may be lost. Consider the following example:

Example 3. — Solve: $5x^2 = 8x$.

Incorrect Solution. — Divide both sides by x: $5x = 8$.

Hence, we obtain only one solution, $x = \frac{8}{5}$.

Correct Solution. — 1. $5x^2 - 8x = 0$.
2. Factor: $x(5x - 8) = 0$.
3. Place each factor equal to zero: $x = 0$, or $5x - 8 = 0$.
4. The solutions are $x = 0$ and $x = \frac{8}{5}$.

Discussion. — In the incorrect solution, the root $x = 0$ was lost.

EXERCISE 44

Solve by factoring, and check the results by substitution:

1. $x^2 + 3x - 10 = 0$.
2. $y^2 + 5y - 14 = 0$.
3. $2x^2 + x - 3 = 0$.
4. $5x^2 - 7x = 0$.
5. $3x^2 = 4x$.
6. $6z^2 - 4z - 10 = 0$.
7. $w^2 - 9 = 0$.
8. $4x^2 - 12x + 9 = 0$.
9. $25z^2 - 4 = 0$.
10. $3w^2 = 11w - 10$.
11. $25y^2 + 20y + 4 = 0$.
12. $8x^2 - 2x - 3 = 0$.
13. $3x^2 + x - 10 = 0$.
14. $3x^2 = 21x$.
15. $x^2 - 144 = 0$.
16. $6x^2 - 11x - 10 = 0$.
17. $5x^2 + 3x = 0$.
18. $2w^2 = 3 - 5w$.
19. $5 + 4z = z^2$.
20. $16z^2 - 1 = 0$.

Solve each equation for x, or for w, by use of factoring:

21. $x^2 - 5ax + 6a^2 = 0$.
22. $3x^2 - bx - 4b^2 = 0$.
23. $2w^2 - 5cw + 3c^2 = 0$.
24. $5x^2 = ax$.
25. $\dfrac{2-x}{3+x} - \dfrac{3x-6}{x-1} = 0$.
26. $\dfrac{8}{w-3} = \dfrac{w-2}{w-4} + 1$.
27. $\dfrac{3}{4x^2} + \dfrac{7}{8x} - \dfrac{5}{2} = 0$.
28. $\dfrac{w+2a}{w+3a} + 1 = \dfrac{36a^2}{(w+3a)^2}$.

80. Completing a square. — A binomial $(x^2 + px)$ becomes a perfect square trinomial if we add **the square of one half of the coefficient of x**, that is, if we add $\left(\frac{p}{2}\right)^2$, or $\frac{p^2}{4}$, because

$$x^2 + px + \frac{p^2}{4} = \left(x + \frac{p}{2}\right)^2 \qquad (1)$$

In adding $\frac{p^2}{4}$ we say that we have **completed a square.**

Thus, to make $(x^2 - 7x)$ a perfect square, we add $(\frac{7}{2})^2$, or $\frac{49}{4}$:

$$x^2 - 7x + \tfrac{49}{4} = (x - \tfrac{7}{2})^2.$$

81. To solve a quadratic equation in x by the method of completing the square, proceed as follows:

1. *Transpose all terms involving x to the left member and all other terms to the right member; collect terms.*
2. *Divide both members by the coefficient of x^2.*
3. *Compute one half of the coefficient of x; square the result, and add this square to both members of the equation.*
4. *Rewrite the left member as the square of a binomial.*
5. *Extract the square roots of both members, using the double sign $\pm$ in the right member.*
6. *Solve each of the two linear equations obtained in Step 5.*

NOTE. — If it is desired to check a solution obtained by completing the square, substitute the roots as given in radical form, because the final decimal values are only approximate solutions and are inconvenient to substitute.

Example 1. — Solve by completing the square:

$$3x^2 - 8x + 2 = 0.$$

Solution. — 1. Subtract 2: $3x^2 - 8x = -2.$

2. Divide by 3: $x^2 - \frac{8}{3}x = -\frac{2}{3}.$

3. Since $\frac{8}{3} \div 2 = \frac{4}{3}$, add $(\frac{4}{3})^2$, or $\frac{16}{9}$, to both members:

$$x^2 - \frac{8}{3}x + \left(\frac{4}{3}\right)^2 = \frac{16}{9} - \frac{2}{3} = \frac{16 - 6}{9}.$$

4. Simplify: $(x - \frac{4}{3})^2 = \frac{10}{9}.$

5. Extract square roots: $x - \frac{4}{3} = \pm\sqrt{\frac{10}{9}} = \pm\frac{\sqrt{10}}{3}.$

6. When $x - \frac{4}{3} = +\frac{\sqrt{10}}{3}$, $x = \frac{4}{3} + \frac{\sqrt{10}}{3} = \frac{4 + \sqrt{10}}{3} = \frac{4 + 3.162}{3}.$

$$x = 2.387.$$

7. When $x - \frac{4}{3} = -\frac{\sqrt{10}}{3}$; $x = \frac{4}{3} - \frac{\sqrt{10}}{3} = \frac{4 - 3.162}{3} = \frac{.838}{3}.$

$$x = .279.$$

EXERCISE 45

Solve by completing the square; find the decimal values of all surd roots; check, when directed by the instructor:

1. $x^2 + 5x - 24 = 0.$
2. $x^2 + 6x - 7 = 0.$
3. $2x^2 - 4x - 126 = 0.$
4. $3x^2 + 9x - 12 = 0.$
5. $z^2 - 10z + 23 = 0.$
6. $x^2 + 3x - 14 = 0.$
7. $w^2 + 9w + 9 = 0.$
8. $w^2 - \frac{4}{3}w = \frac{4}{3}.$
9. $x^2 + \frac{5}{6}x - \frac{25}{6} = 0.$
10. $6x^2 - x - 15 = 0.$
11. $5 = 6x^2 + 13x.$
12. $8x^2 - 3 = 2x.$
13. $2w^2 - 10w + 5 = 0.$
14. $3x^2 + 3 = -10x.$
15. $2y^2 - 5y - 1 = 0.$
16. $5y^2 - 12y + 3 = 0.$
17. $3y^2 + 6y - 2 = 0.$
18. $3m^2 + 5m - 3 = 0.$
19. $4y^2 + 7y - 1 = 0.$
20. $5x^2 - 4x - 2 = 0.$
21. $2y^2 - 8y + 5 = 0.$
22. $7x^2 - 9x + 1 = 0.$
23. $2w^2 + 11w + 5 = 0.$
24. $3h^2 - 5h - 4 = 0.$

25. $10\,w^2 + 4\,w - 3 = 0.$ **28.** $5\,x^2 - 3\,x - 2 = 0.$

26. $6\,x^2 + 8\,x - 5 = 0.$ **29.** $.5\,w^2 + .3\,w - .08 = 0.$

27. $.3\,x^2 - .06\,x - .144 = 0.$ **30.** $x^2 + .4\,x - .5 = 0.$

Solve the following equations for x, by completing the square:

31. $ax^2 + hx + k = 0.$

Solution. — 1. Subtract k: $ax^2 + hx = -k.$

2. Divide by a: $x^2 + \frac{h}{a}x = -\frac{k}{a}.$

3. Add $\left(\frac{h}{2a}\right)^2$ or, $\frac{h^2}{4a^2}$: $x^2 + \frac{h}{a}x + \left(\frac{h}{2a}\right)^2 = \frac{h^2}{4a^2} - \frac{k}{a}.$

4. $\left(x + \frac{h}{2a}\right)^2 = \frac{h^2 - 4ka}{4a^2}.$

5. Extract square roots: $x + \frac{h}{2a} = \pm\sqrt{\frac{h^2 - 4ka}{4a^2}}.$

6. Subtract $\frac{h}{2a}$: $x = -\frac{h}{2a} \pm \frac{\sqrt{h^2 - 4ka}}{2a} = \frac{-h \pm \sqrt{h^2 - 4ka}}{2a}.$

32. $x^2 + 2\,ax = 3\,a^2.$ **38.** $Ax^2 + Bx + C = 0.$

33. $x^2 + 2\,bx + c = 0.$ **39.** $x^2 + 3\,x = m^2 - 3\,m.$

34. $ax^2 + 2\,x - b = 0.$ **40.** $x^2 + 2\,ax = 1 + 2\,a.$

35. $2\,x^2 - 8\,x - b = 0.$ **41.** $x^2 - (a - b)x = ab.$

36. $ax^2 + bx - 5 = 0.$ **42.** $ax^2 + bx - c = 0.$

37. $ax^2 + 4\,x + c = 0.$ **43.** $ax^2 - (a^2 + b)x + ab = 0.$

Verify the statement by substitution in the equation:

44. $x = \frac{2 + \sqrt{2}}{2}$ and $x = \frac{2 - \sqrt{2}}{2}$ are solutions of $2\,x^2 - 4\,x + 1 = 0.$

45. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ are solutions of $ax^2 + bx + c = 0.$

82. The quadratic formula. — Every quadratic equation in a single variable can be reduced to the standard form

$$ax^2 + bx + c = 0, \qquad (1)$$

where a, b, and c are constants and $a \neq 0$. We can solve (1) by the method of completing the square.

Solution of $ax^2 + bx + c = 0$.

1. Subtract c: $\quad ax^2 + bx = -c.$
2. Divide by a: $\quad x^2 + \frac{b}{a}x = -\frac{c}{a}.$
3. Add $\left(\frac{b}{2a}\right)^2$ or, $\frac{b^2}{4a^2}$: $\quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}.$
4. Simplify: $\quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$
5. Extract square roots: $\quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}.$
6. Subtract $\frac{b}{2a}$: $\quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$
7. Hence, if $ax^2 + bx + c = 0$ has *any* solutions, they are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (2)$$

Formula 2 is called the **quadratic formula**. It should be memorized.

In Problem 45, Exercise 45, it was shown, by substitution, that the values of x in (2) actually satisfy (1). Hence, we have proved that *every quadratic equation in one variable has two and only two roots, which are given by the quadratic formula.*

To solve a quadratic equation by use of the quadratic formula, *reduce the equation to the standard form* $ax^2 + bx + c = 0$, *and then substitute the values of* a, b, *and* c *in formula* 2.

Example 1. — Solve by use of the quadratic formula:

$$3x^2 - 6x = 2.$$

Solution. — 1. Reduce to standard form: $3x^2 - 6x - 2 = 0$.

2. $a = 3$; $b = -6$; $c = -2$; on substituting in (2) we obtain

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 3 \cdot (-2)}}{6} = \frac{6 \pm 2\sqrt{15}}{6} = \frac{3 \pm (3.873)}{3}.$$

3. Hence, $x = \frac{3 + 3.873}{3} = 2.291$, and $x = \frac{3 - 3.873}{3} = -.291$.

NOTE. — In advanced mathematics, it is proved that *an integral rational equation of degree n has exactly n roots.* In deriving the quadratic formula, we have proved that this theorem is true when $n = 2$.

EXERCISE 46

Solve by use of the quadratic formula:

1. $6x^2 + 13x + 6 = 0$.
2. $2x^2 + x - 6 = 0$.
3. $5x^2 + 7x - 6 = 0$.
4. $2w^2 + 5w - 12 = 0$.
5. $3y^2 + 7y + 2 = 0$.
6. $x^2 + 6x + 6 = 0$.
7. $x^2 + 3x + 1 = 0$.
8. $2x^2 + x - 2 = 0$.
9. $5x^2 - 3x - 2 = 0$.
10. $y^2 - .5y + .04 = 0$.
11. $z^2 - 8z - 2 = 0$.
12. $3w^2 - 8w + 2 = 0$.
13. $-x^2 + x + 3 = 0$.
14. $2x + 2 = 3x^2$.
15. $4x^2 = 5 + 2x$.
16. $5 = 3x + 2x^2$.
17. $2x^2 - 5 = 0$.
18. $0 = 2 - 4x - x^2$.
19. $2y^2 - .3y - .2 = 0$.
20. $5x^2 - x - 1 = 0$.
21. $6z^2 = 7z + 3$.
22. $3x^2 + 5x - 4 = 0$.
23. $2w^2 + .3w - .1 = 0$.
24. $w^2 - 3w - 1 = 0$.
25. $0 = 3x^2 - 2x - 5$.
26. $.5 = x^2 - 2x$.
27. $3 - 3x = 2x^2$.
28. $y^2 + .6y - .4 = 0$.
29. $h^2 + .3h - .4 = 0$.
30. $8 = 2x^2 - 5x$.

Solve for x (or for y) by use of the quadratic formula:

31. $2x^2 - 2dx - 5d^2 = 0$.

HINT. — Substitute $a = 2$, $b = -2d$, and $c = -5d^2$.

32. $r^2x^2 - rx - 3 = 0$.
33. $x^2 + 2ax - 3d = 0$.
34. $2x^2 + 3ax + 4 = 0$.
35. $cy^2 + 2by + 3 = 0$.
36. $x^2 - (2c + d)x + 2cd = 0$.
37. $x^2 + ax + x + a = 0$.
38. $cx^2 + bx - a = 0$.
39. $ay^2 - by - c = 0$.

83. Simplification of imaginary numbers. — If A is negative, we have agreed that $\sqrt{A}$ represents a new kind of number which we call an *imaginary** number, and, for contrast, we call positive and negative numbers real numbers.

Let $i = \sqrt{-1}$; then $i^2 = -1$ and the square roots of -1 are $\pm\sqrt{-1} = \pm i$. Moreover, if P is any positive number, $\sqrt{-P}$ can be expressed as a product of a real number and i.

Thus, the square roots of -5 are

$$\pm\sqrt{-5} = \pm\sqrt{(-1)\cdot 5} = \pm\sqrt{-1}\sqrt{5} = \pm i\sqrt{5}.$$

If a and b are any real numbers, then $(a + bi)$ is called a **complex number**, whose real part is a and imaginary part is bi. If $b \neq 0$, then $(a + bi)$ is called an **imaginary number**. An imaginary number is called a **pure imaginary** in case its real part is zero.

Thus, $(2 + 7i)$ is an imaginary number; $5i$ is a pure imaginary number.

NOTE. — The name of imaginary numbers is inherited from a time when these numbers actually appeared imaginary to mathematicians. For a similar reason, negative numbers were called fictitious numbers by the early mathematicians.

Unless otherwise stated all literal numbers in this book represent real numbers.

To simplify an indicated square root of a radicand bearing a minus sign, express the radical as the product of $\sqrt{-1}$ and a radical whose radicand has a plus sign, and introduce i for $\sqrt{-1}$.

Thus, $$\sqrt{-\frac{9a^4}{2}} = \sqrt{(-1)\cdot\frac{9a^4}{2}} = \sqrt{-1}\sqrt{\frac{9a^4}{2}}$$
$$= i\cdot\frac{3a^2}{2}\sqrt{2} = \frac{3a^2 i}{2}\sqrt{2}.$$

* Read Section 55, page 72.

EXERCISE 47

Simplify each radical and express by use of i:

1. $\sqrt{-9}$.	**6.** $\sqrt{-169}$.	**11.** $\sqrt{-128\,x^6}$.	**16.** $\sqrt{-.04}$.
2. $\sqrt{-25}$.	**7.** $\sqrt{-8}$.	**12.** $\sqrt{-200}$.	**17.** $\sqrt{-\frac{4}{25}}$.
3. $\sqrt{-144}$.	**8.** $\sqrt{-98}$.	**13.** $\sqrt{-\frac{1}{4}}$.	**18.** $\sqrt{-\frac{5}{16}}$.
4. $\sqrt{-5}$.	**9.** $\sqrt{-125\,a^2}$.	**14.** $\sqrt{-\frac{1}{9}}$.	**19.** $\sqrt{-\frac{81}{49}}$.
5. $\sqrt{-49\,x^2}$.	**10.** $\sqrt{-132}$.	**15.** $\sqrt{-150}$.	**20.** $\sqrt{-\frac{18}{7}}$.

21. $\dfrac{3+\sqrt{-18}}{2}$. **22.** $\dfrac{6-\sqrt{-50}}{4}$. **23.** $\dfrac{-10+\sqrt{-150}}{5}$.

24. By use of the fact that $i^2 = -1$, prove that $i^3 = -i$; $i^4 = 1$; $i^5 = i$.

Perform indicated operations and simplify by use of the fact that $i^2 = -1$; $i^5 = i$; $i^4 = 1$; etc.:

25. $(3\,i-2)(i+3)$. **28.** $(3+2\,i)^2$.

26. $(4\,i+2)(2-3\,i)$. **29.** $\sqrt{-3}\,(\sqrt{-5}+\sqrt{-3})$.

27. $(i^3-2\,i)(3\,i+5)$. **30.** $(i^4-3\,i)(i+2)$.

31. Find the two square roots of each number:

$$-25;\quad -64;\quad -15;\quad -\tfrac{1}{8}.$$

84. Quadratic equations whose roots are imaginary.

Example 1. — Solve by use of the quadratic formula:

$$2x^2 - 4x + 5 = 0.$$

Solution. — 1. $a = 2$, $b = -4$, $c = 5$: $\quad x = \dfrac{4 \pm \sqrt{16-40}}{4}$.

2. Since $\sqrt{-24} = 2i\sqrt{6}$, $\quad x = \dfrac{4 \pm 2i\sqrt{6}}{4} = \dfrac{2 \pm i\sqrt{6}}{2}$.

NOTE. — If an imaginary number occurs in the solution of a quadratic, simplify the number by introducing i; do not insert the values of indicated square roots of real numbers, unless so directed.

EXERCISE 48

1. By substitution in the equation, show that $x = (1 + i)$ and $x = (1 - i)$ are roots of $(x^2 - 2x + 2) = 0$.

Solve by use of the quadratic formula:

2. $x^2 + 2x + 2 = 0$.
3. $w^2 - 2w + 5 = 0$.
4. $2y^2 + 2y + 3 = 0$.
5. $6z^2 - 3z + 5 = 0$.
6. $3z^2 + 4z + 8 = 0$.
7. $2x^2 - 4x + 20 = 0$.
8. $4x^2 - 8x + 9 = 0$.
9. $5x^2 - 2x + 3 = 0$.

Solve by the method of completing the square:

10. $2x^2 + 4x + 6 = 0$.
11. $3x^2 - 6x + 5 = 0$.

85. Summary. — A quadratic equation should be solved by factoring, if factors can be *easily* recognized; otherwise the equation should be solved by use of the quadratic formula.

EXERCISE 49

Solve (a) by factoring, (b) by completing the square, and (c) by use of the formula:

1. $2x^2 + 5x - 3 = 0$.
2. $3x^2 - 16x + 5 = 0$.

Solve by the most convenient method:

3. $x^2 - 3x + 2 = 0$.
4. $2w^2 - 7 = 0$.
5. $3x^2 - 5x - 6 = 0$.
6. $5x^2 = 9x$.
7. $5x^2 - 2x + 3 = 0$.
8. $2x^2 + 5x + 3 = 0$.
9. $8z^2 - 5z - 2 = 0$.
10. $3 = 5z - 4z^2$.
11. $5y^2 + 9y + 3 = 0$.
12. $5x^2 - 3 = 2x^2 + 7$.
13. $2y^2 + 3y + 8 = 0$.
14. $x^2 - 8x = 0$.
15. $\frac{8}{x} - \frac{5}{x+3} = 3$.
16. $\frac{2}{x-1} - \frac{1}{3x+1} = \frac{1}{4}$.
17. $\frac{3x}{x-2} - \frac{1}{x^2-4} = 2$.
18. $\frac{3x+9}{x+1} - \frac{2x-1}{x-1} = 2$.

19. $\frac{3x+5}{3x+2}+\frac{2x+1}{2x}=1.$

20. $\frac{2w}{w+5}-\frac{4}{w-3}=1.$

21. $\frac{7}{1-x}+\frac{5x}{x^2-3x+2}=\frac{1}{3}.$

22. $\frac{2c-1}{3c-5}=\frac{3c+1}{4c}-\frac{1}{6}.$

Solve for x:

23. $\frac{1}{2}gx^2+ax=S.$

24. $kx^2+2kx+x+2=0.$

25. $4x^2+2bx+(b-1)=0.$

26. $x^2+hx+h-1=0.$

Solve each problem by introducing only one unknown number; check the solution by substitution in the problem:

27. If 4 is subtracted from five times a certain number, the result equals the square of the number. Find the number.

28. Find two consecutive integers whose product is 42.

29. The area of a certain rectangle is 368 square feet. One dimension is 7 feet longer than the other. Find the dimensions.

30. If 38 is divided into two parts whose product is 357, find the two parts.

31. In a right-angled triangle, the square of the hypothenuse equals the sum of the squares of the other two sides. In a certain right-angled triangle, the hypothenuse is $\sqrt{53}$ feet long. The sum of the lengths of the other two sides is 9 feet. Find the length of these two sides.

32. The diagonal of a certain square is 3 feet longer than a side of the square. Find the length of a side.

33. If the sum of a certain number and its reciprocal is $\frac{13}{6}$, find the number.

34. The sum of a certain number and twice its reciprocal is $\frac{73}{8}$. Find the number.

35. The side of one square is 3 feet longer than the side of another. The sum of the areas of the two squares is 89 square feet. Find the lengths of their sides.

36. The denominator of a certain fraction exceeds the numerator by 1. If 1 is added to both numerator and denominator,

the new fraction exceeds the original fraction by $\frac{1}{12}$. Find the original fraction.

37. A certain rectangular lawn is 40 yards long and 30 yards wide. The lawn is increased to a larger rectangle, of double the original area, by adding a border of uniform width. How wide is the border?

38. A certain rectangular field is 40 rods long and 20 rods wide. A man has plowed a uniform border inside the field, and only $\frac{1}{3}$ of the plowing of the field remains to be done. How wide is the border?

CHAPTER X

FUNCTIONS AND THEIR GRAPHS

86. Related variables.* — To say that a variable y is a *function* of another variable x means that for every value of the independent variable x we can determine a corresponding value for y.

If two variables are related by an equation, then either may be thought of as a function of the other.

Thus, in the equation $(2x - y) = 3$, we can say that y is a function of x whose values are given by the formula $(2x - 3)$, because on solving the equation for y we obtain $y = 2x - 3$. Similarly, we can say that x is a function of y whose values are given by the formula $x = \frac{1}{2}(3 + y)$.

87. Graph of a function. — Let y represent any function of x. Then, each pair of corresponding values of x and y can be taken as the coördinates of a point, in a system of coördinates having an x-axis and a y-axis.

The graph of a function is the curve (or curves) which

1. *contains all points whose coördinates (x, y) form a pair of corresponding values of x and the function of x, and*
2. *contains no other points.*

In drawing the graph of a function it is customary to plot the values of the independent variable on the horizontal axis and values of the function, or dependent variable, on the vertical axis.

* Review the definitions of the terms constant, variable, function of a variable, and graph of an equation, in Chapter V, Sections 36 to 41.

Example 1. — Graph the function $(x^2 - x - 6)$.

Solution. — 1. Let $y = x^2 - x - 6$.

2.

When x =	-4	-3	-1	0	$\frac{1}{2}$	1	2	4	5
then y =	14	6	-4	-6	$-6\frac{1}{4}$	-6	-4	6	14

3. Plot the points whose coördinates are the pairs of values of x and y in the table of Step 2. The smooth curve drawn through the plotted points, in Figure 6, is the graph of the function.

The curve obtained in Example 1 is called a **parabola**. The rounded end of a parabola is called its **vertex**.

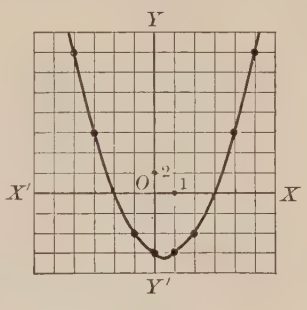


Fig. 6.

In Example 1, notice that the graph of $(x^2 - x - 6)$ is the graph of the equation $y = x^2 - x - 6$, because both graphs are determined by the corresponding values of x and y, where y is thought of merely as an abbreviation for the function $(x^2 - x - 6)$. In the same way, it is seen that **the graph of any function of x is the graph of the equation obtained on placing y equal to the function.**

Thus, to graph the function $(5x^3 - 3)$, we graph the equation $y = 5x^3 - 3$.

Note. — One of the most useful purposes of a graph of a function is to exhibit clearly and quickly to the eye the corresponding values of the independent variable and the function. Graphs of functions also have many other uses, some of which will be met later in this book.

88. **A linear function** of x is a polynomial of the first degree in x, and hence has the form $(ax + b)$, where a and b are constants. **The graph of a linear function $(ax + b)$ is a straight line**, because it is the graph of the linear equation $y = ax + b$.

89. A quadratic function of x is a polynomial of the second degree in x, and hence has the form $(ax^2 + bx + c)$, where a, b, and c are constants. The following facts are proved in advanced mathematics.

1. The graph of a quadratic function $(ax^2 + bx + c)$ is a parabola which is open upward if a is positive, and downward if a is negative.

2. The abscissa of the vertex of the parabola is $x = -\frac{b}{2a}$.

Example 1. — Graph the function $(-3x^2 + 6x + 7)$.

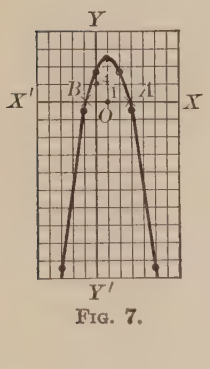


FIG. 7.

Solution. — 1. Let $y = -3x^2 + 6x + 7$.

2. The graph will be a parabola, open downward; the abscissa of the vertex (which will be the highest point) is $x = \frac{6}{6} = 1$.

WHEN x =	-3	-1	$\frac{1}{2}$	1	$\frac{3}{2}$	3	5
THEN y =	-38	-2	$9\frac{1}{4}$	10	$9\frac{1}{4}$	2	-38

3. The graph is given in Figure 7.

Notice that, in the table of values, we chose values of x symmetrically about $x = 1$, the abscissa of the vertex.

EXERCISE 50

In graphing quadratic functions in this exercise, use values of x extending 5 units on each side of the abscissa of the vertex.

1. Figure 7 on page 114 is the graph of the function $(-3x^2 + 6x + 7)$. By measuring the ordinates of the proper points on that graph, find the value of the function corresponding to (*a*) $x = -2$; (*b*) $x = -\frac{1}{2}$; (*c*) $x = 2\frac{1}{2}$.

HINT for Part *a*. — Measure y at the point where $x = -2$.

2. In Figure 7, find the value of x for which the corresponding value of y is (*a*) $y = 4$; (*b*) $y = -8$; (*c*) $y = 0$.

Graph the following functions:

3. $3x - 2$.	**7.** $x^2 - x - 4$.	**11.** $2x^2 - 6x - 9$.
4. $5x + 4$.	**8.** $-x^2 + x + 4$.	**12.** $x^2 - 5x + 3$.
5. $6x - 11$.	**9.** $x^2 - 4x + 5$.	**13.** $3x^2 - 9x$.
6. $8x^2 - 5$.	**10.** $-x^2 - 6x + 12$.	**14.** $-2x^2 + 10x - 4$.

Graph the following equations:

15. $y - 3x + 4 = 0$. **17.** $2y - 8x^2 - 3 = 0$.

16. $x^2 - 2x + 3 = y$. **18.** $y - 5x + 3x^2 - 2 = 0$.

19. Graph the following functions on the same coördinate system, using the same values of x in the three tables of values: $(x^2 - 2x - 5)$; $(x^2 - 2x + 1)$; $(x^2 - 2x + 7)$.

20. How is the graph of $(ax^2 + bx + c)$ related to the graph of $(ax^2 + bx + c + 10)$, in the same coördinate system? How is the graph of $(ax^2 + bx + c)$ related to the graph of $(ax^2 + bx + d)$, (I) if $d > c$; (II) if $d < c$.

90. Graphical solution of an equation in one variable, x:

1. *Simplify and transpose all terms to the left member, obtaining an equation of the form $F = 0$, where F is some function of x.*

2. *Draw a graph of F; that is, graph the equation $y = F$.*

3. *Find the abscissas of the points where the graph cuts the x-axis. These abscissas are values of x for which the corresponding value of the function F is zero; that is, these abscissas are roots of the equation $F = 0$.*

Example 1. — Solve graphically: $-3x^2 + 6x + 7 = 0$.

Solution. — 1. Graph the function $y = -3x^2 + 6x + 7$. The graph is the one constructed in Figure 7, page 114.

2. Notice that the graph cuts the x-axis at points A and B. The abscissa of A is $x = 2.8$; the abscissa of B is $x = -.8$. Each of these is a value of x for which the corresponding value of $(-3x^2 + 6x + 7)$ is zero. Hence, the abscissas of A and B are roots of the equation $(-3x^2 + 6x + 7) = 0$.

Check. — Solving algebraically, we obtain $x = 2.826$ and $x = -.826$.

NOTE. — By a graphical solution of an equation, we can find only the real solutions of the equation, because the solutions obtained by use of a graph are the abscissas of certain points, and hence are real numbers. Thus, if the roots of a quadratic equation are imaginary numbers, we cannot obtain the roots graphically.

91. The graphical solution of a quadratic equation

$$ax^2 + bx + c = 0, \tag{1}$$

is performed by use of the graph of the function

$$y = ax^2 + bx + c.$$

NOTE. — The parabola, which is the graph of $(ax^2 + bx + c)$,

I: *will not meet the x-axis, when and only when* (1) *has imaginary roots.*

II: *will touch the x-axis in only one point, that is, will be tangent to the x-axis, when and only when* (1) *has equal roots;* and

III: *will cut the x-axis in two points, when and only when* (1) *has two unequal real roots,*

The three graphical possibilities are illustrated in Figure 8 where the parabolas I, II, and III are, respectively, the graphs of the functions in the left members of the following equations I, II, and III:

I: $x^2 - 2x + 5 = 0.$

II: $x^2 - 2x + 1 = 0.$

III: $x^2 - 2x - 8 = 0.$

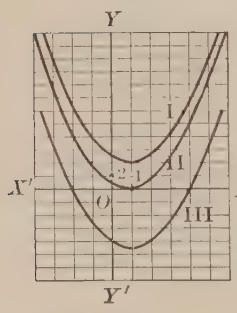


FIG. 8.

From the graphs, we see that equation I has imaginary roots; equation II has two equal roots, $x = 1$; equation III has the roots $x = 4$ and $x = -2$.

NOTE. — For convenience in solving a quadratic equation graphically, make the coefficient of x positive before drawing a graph. Then, the parabola involved will always be open upward. Thus, before solving $(-2x^2 - x + 3) = 0$ we change it to $(2x^2 + x - 3) = 0$.

EXERCISE 51

Solve each equation graphically:

1. $2x - 9 = 0$.
2. $x^2 - 4 = 0$.
3. $3x = 7$.
4. $x^2 - 6x - 7 = 0$.
5. $x^2 - 3x + 2 = 0$.
6. $x^2 - 4x + 4 = 0$.
7. $x^2 - 4x + 8 = 0$.
8. $9 = 5x - x^2$.
9. $2x^2 + 10 = 9x$.
10. $6 + 5x = x^2$.
11. $3x - 8x^2 = 0$.
12. $-2x^2 + 3x - 4 = 0$.
13. $4x^2 - 12x + 9 = 0$.
14. $2x^2 + x = 10$.
15. $2z^2 - 3z - 4 = 0$.
16. $2 = x^2 - 5x^2$.
17. $y^2 - 2y = \frac{2}{3}$.
18. $\frac{1}{2}y^2 + 2 + 2y = 0$.
19. $\frac{2}{5}x^2 - \frac{1}{4}x - 2 = 0$.
20. $w^2 + 2w - 4 = 0$.

92. Functions defined by tables of values. — In applications of graphs of functions, it is often found that the function involved is neither given by a formula nor defined by an equation. In some cases, all information concerning the function is represented by a table of pairs of corresponding values of the function and the independent variable. In drawing the graph of such a function, *draw a smooth curve through the points obtained from the table of values*, unless otherwise directed.

EXERCISE 52

1. The following table gives the time it takes money to double itself if invested at certain rates of interest, compounded annually. Graph the time as a function of the rate.

TIME, YEARS	70	$46\frac{1}{2}$	35	28	$23\frac{1}{2}$	$17\frac{1}{2}$	14	12
RATE . . .	1%	$1\frac{1}{2}$%	2%	$2\frac{1}{2}$%	3%	4%	5%	6%

HINT. — The *rate* is specified as the *independent variable*. Hence, plot *rates* on the horizontal axis.

2. By use of the graph in Problem 1, find the time it will take for money to double itself if the rate is (*a*) $3\frac{1}{2}\%$; (*b*) $5\frac{1}{2}\%$.

3. The velocity of sound in air depends on the temperature of the air. Graph the velocity as a function of the temperature by use of the following data.

Velocity, ft. per sec.	1030	1040	1060	1080	1110	1140	1170
Temp. (Fahrenheit).	− 30°	− 20°	0°	20°	50°	80°	110°

Hint. — It is not necessary that the intersection of the axes should be the zero on both scales. In Problem 3, let the intersection of the axes represent 1000 feet on the vertical axis.

4. From the graph in Problem 3, read the velocity of sound when the temperature is (*a*) 30°; (*b*) 75°.

5. The following table gives the international amateur records, up to July 16, 1924, for runs of the indicated distances. Graph the record time, in seconds, as a function of the distance.

Distance . .	100 *yd.*	220 *yd.*	440 *yd.*	880 *yd.*	1320 *yd.*	1760 *yd.*
Record time	9.6 *s.*	20.8 *s.*	47.4 *s.*	1 *m.* 52.2 *s.*	3 *m.* 2.8 *s.*	4 *m.* 10.4 *s.*

6. From the graph in Problem 5, estimate the record time which should correspond to a 1000 yard run.

7. Construct a graph of the function $\sqrt{x}$ for values of x from $x = 0$ to $x = 10$.

Hint. — Make a large scale graph and, in your table of values, use the following values of x: 0; $\frac{1}{8}$; $\frac{1}{4}$; 1; 2; 3; 4; 5; 7; 10.

8. From the graph in Problem 7, read the values of the following square roots: $\sqrt{1.5}$; $\sqrt{2.75}$; $\sqrt{3.5}$; $\sqrt{3.25}$.

The average weights and heights of boys of certain ages in the United States are given in the following table. The heights are correct to the nearest inch, and the weights are correct to the nearest pound.

Age, years	1	2	3	4	6	8	10	13	16
Weight . .	22	27	32	36	45	54	65	85	121
Height . .	29	34	37	39	44	48	52	57	65

By use of the preceding table, graph the specified functions:

9. The weight of boys as a function of their age.

10. The height of boys as a function of their age.

11. From the graphs in Problems 9 and 10 determine the average weight and height of boys whose age is 5 years; 12 years; 15 years.

12. The weight of a cubic foot of dry air at an atmospheric pressure of 29.92 inches of mercury, under various temperatures, is given in the following table, where weight is in pounds, and temperature is in degrees Fahrenheit. By use of the table, graph the weight of air as a function of the temperature.

Temperature .	0°	12°	32°	52°	82°	112°	152°	192°	212°
Weight . . .	.0864	.0842	.0807	.0776	.0733	.0694	.0646	.0609	.0591

13. From the graph in Problem 12, find the weight of a cubic foot of dry air, under the given pressure, at a temperature of 90°; 130°; 200°.

93. Functional notation. — Any convenient letter may be introduced to represent a function. Thus, we may let $f(x)$ represent $(3x^2 - 5)$. "$f(x)$" is read "*the f function of x*," or, for short, merely "*f of x*." Similarly, we may let $H(y) = (3y + 5)$; this equation is read "*H of y equals* $(3y + 5)$," and we place y in parentheses after H to indicate the variable of which H is a function.

Let $F(x)$ represent a function of x; then, if a is any value of x,

$$F(a) \text{ represents the value of } F(x) \text{ when } x = a. \qquad (1)$$

The symbol "$F(a)$" is read "*F of a*." If $F(x) = (3x^2 - 5 - x)$,

$$F(3) = 3 \cdot 3^2 - 5 - 3 = 19;$$
$$F(-3) = 3(-3)^2 - 5 + 3 = 25;$$
$$F(-b^2) = 3(-b^2)^2 - 5 - (-b^2) = 3b^4 - 5 + b^2.$$

EXERCISE 53

Read each of the following symbols in words:

1. $f(w)$. **2.** $F(z)$. **3.** $k(x)$. **4.** $F(x)$. **5.** $H(u)$.

If $f(x) = (3x - 2)$, *tell in words what each of the following symbols represents, and find the value of each one:*

6. $f(4)$. **7.** $f(-3)$. **8.** $f(\frac{1}{3})$. **9.** $f(2a)$. **10.** $f\left(\frac{1}{b}\right)$.

11. If $G(y) = 3y - 2y^2$, find $G(1)$; $G(-3)$; $G(\frac{1}{4})$.

12. If $F(x) = x^2$, find $F(0)$; $F(2)$; $3F(-5)$; $aF\left(\frac{1}{a}\right)$.

13. If $f(u) = \dfrac{u+1}{u}$, find $f(1)$; $5f(-3)$; $[f(b)]^2$; $f(a-1)$.

14. If $K(z)$ represents $\left(3z - \dfrac{1}{z}\right)$, find $K(2)$; $[K(-2)]^2$; $K(\frac{1}{3})$; $K(b)$.

15. If $f(x) = x^3 - 4x^2 - 2x + 8$, fill in the table:

IF x =	-3	-2	-1	$-\frac{1}{2}$	0	1	2	$2\frac{1}{2}$	3	4	5
THEN $f(x)$ =											

16. Graph the function $f(x)$ of Problem 15 by use of the table of values in Problem 15, and then solve the following equation graphically: $x^3 - 4x^2 - 2x + 8 = 0$.

CHAPTER XI

ADVANCED TOPICS IN QUADRATIC EQUATIONS

94. Character of the roots. — Let r and s represent the roots of $(ax^2 + bx + c) = 0$: then, by the quadratic formula,

$$(1): r = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \quad (2): s = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We assume that a, b, and c are real numbers and $a \neq 0$. Then, r and s are real unless $(b^2 - 4ac)$ is negative, and whenever $(b^2 - 4ac)$ is negative both r and s are imaginary. If $(b^2 - 4ac) = 0$, then $r = s = -\frac{b}{2a}$, or the roots are real and equal. Moreover, whenever $r = s$, then, on subtracting each side of (2) from the corresponding side of (1), we obtain

$$0 = \frac{2\sqrt{b^2 - 4ac}}{2a}, \quad \text{or} \quad \sqrt{b^2 - 4ac} = 0,$$

and hence $(b^2 - 4ac) = 0$.

SUMMARY

The Roots of $(ax^2 + bx + c) = 0$ are	When and Only When
real and equal	$b^2 - 4ac = 0$
unequal real numbers	$b^2 - 4ac > 0$
imaginary numbers	$b^2 - 4ac < 0$

We call $(b^2 - 4ac)$ the **discriminant** of the equation $(ax^2 + bx + c) = 0$, because, as soon as we know the value of

$(b^2 - 4ac)$, we can tell the general character of the roots of the equation without actually solving it.

If a, b, and c are rational* numbers, the roots are rational when and only when $\sqrt{b^2 - 4ac}$ is real and is a rational number; that is, *the roots are rational when and only when the discriminant is positive or zero, and is a perfect square.*

ILLUSTRATIONS OF THE USE OF THE DISCRIMINANT

EQUATION	DISCRIMINANT	HENCE, THE ROOTS ARE
$4x^2 - 3x + 5 = 0$	$(-3)^2 - 4 \cdot 4 \cdot 5 = -71$	*imaginary numbers*
$4x^2 - 4x + 1 = 0$	$4^2 - 4 \cdot 4 = 0$	*real; equal; rational*
$4x^2 - 3x - 5 = 0$	$(-3)^2 + 4 \cdot 4 \cdot 5 = 89$	*real; unequal; irrational*
$x^2 - 2x - 3 = 0$	$(-2)^2 - 4(-3) = 16 = 4^2$	*real; unequal; rational*

EXERCISE 54

Compute the discriminant and tell the character of the roots, without solving:

1. $y^2 - 8y + 12 = 0$.
2. $9x^2 - 24x + 16 = 0$.
3. $2x^2 + 5x - 3 = 0$.
4. $4x^2 - 12x + 9 = 0$.
5. $3x^2 - 5x - 9 = 0$.
6. $4x^2 + 5 = -8x$.
7. $6x = 3x^2 + 7$.
8. $3y^2 - 2y = 5$.
9. $3y^2 - 5y + 3 = 0$.
10. $\frac{1}{3}x^2 - 3x = 2$.
11. $x^2 - .5x + .4 = 0$.
12. $3w^2 - 2.1w + .2 = 0$.
13. $3x = 5x^2 - 5$.
14. $25z^2 - 30z + 9 = 0$.

(a) Solve graphically; (b) check the graph by computing the discriminant and thus determining the character of the roots:

15. $x^2 - 4x = 5$. 16. $x^2 + 4 = 4x$. 17. $x^2 - 4x + 7 = 0$.

18. What can you say concerning the graph of the function $(ax^2 + bx + c)$ if you know that the discriminant of $(ax^2 + bx + c) = 0$ is (1) positive; (2) negative; (3) zero? Give a reason for each answer.

* See definition of rational numbers, page 82.

SUPPLEMENTARY PROBLEMS

Find the values of the constant k for which the equation will have equal roots:

Example. — Given equation: $kx^2 + 2x^2 - 3kx + k = 0$.

Solution. — 1. Reduce to normal form: $(k + 2)x^2 - 3kx + k = 0$.

2. If the roots are equal, the discriminant is zero. The discriminant is $[(-3k)^2 - 4k(k + 2)]$, or $(5k^2 - 8k)$. Hence,

$$5k^2 - 8k = 0;\ k(5k - 8) = 0;\ k = 0 \text{ or } k = \tfrac{8}{5}.$$

19. $x^2 - 2kx + 4 = 0$.

20. $3x^2 - 5x + k = 0$.

21. $2x^2 - 4x + k + 2 = 0$.

22. $3x^2 - 2kx + k = 0$.

23. $x^2 - 3kx + 2 - k = 0$.

24. $x^2 - kx - x - k = 0$.

25. $k^2x^2 - 3x - 1 = 0$.

26. $x^2 - kx^2 + 5kx - 3k = 0$.

Find the values of the constant k for which the graph of the function will be tangent to the x-axis:

27. $2x^2 - 3kx - k$.

28. $kx^2 + 7x^2 - 3kx + 1$.

29. (*a*) Find a value of k for which the graph of the function $f(x) = (2x^2 - 4kx + 5)$ will not cut the x-axis. (*b*) Substitute this value of k in $f(x)$ and graph the resulting function.

95. The sum and the product of the roots. — From Section 94, we obtain

$$r + s = \frac{-2b}{2a} = -\frac{b}{a};$$

$$rs = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

$$rs = \frac{(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Hence, we have proved the following facts concerning the roots of $ax^2 + bx + c = 0$:

The sum of the roots equals $-\frac{b}{a}$: $\quad r + s = -\frac{b}{a}.$ (1)

The product of the roots equals $\frac{c}{a}$: $rs = \frac{c}{a}$. (2)

Thus, to find the sum and the product of the roots of $3x^2 = (5x - 7)$, first write the equation in the standard form: $3x^2 - 5x + 7 = 0$. The sum of the roots is $\frac{5}{3}$ and their product is $\frac{7}{3}$.

EXERCISE 55

Find the sum and the product of the roots without solving:

1. $x^2 - 5x + 6 = 0$.
2. $9x^2 + 12x + 4 = 0$.
3. $x^2 - 4x + 6 = 0$.
4. $w^2 - 9w + 4 = 0$.
5. $16m^2 - 4 = 0$.
6. $x^2 + .3 = .5x$.
7. $30x = 9x^2 + 25$.
8. $9x^2 = 3x - 1$.
9. $2z^2 + 2 = 5z$.
10. $24w = 9w^2 + 16$.

11. One root of $2x^2 - 5x + 3 = 0$ is 1; find the other root.

HINT. — If s is the other root, then $s + 1 = \frac{5}{2}$.

12. One root of $2x^2 - 7x + 3 = 0$ is 3; find the other root.

Find the value of the constant h, under the given condition:

Example. — The sum of the roots is 5:

$$3hx^2 + 6x - 3hx + 4 = 0.$$

Solution. — 1. Reduce to standard form:

$$3hx^2 + x(6 - 3h) + 4 = 0.$$

2. Since the sum of the roots is 5, hence $5 = -\frac{6 - 3h}{3h}$. Solving this equation, we obtain $h = -\frac{1}{2}$.

13. The sum of the roots is -4: $2hx^2 + 3x - 5hx - 4 = 0$.
14. The product of the roots is 2: $5x^2 - 3x + 4h - 2 = 0$.
15. The product of the roots is $\frac{3}{4}$: $x^2 - 5x + 3h = 0$.
16. The sum of the roots is 4: $2hx^2 + x - hx - 5 = 0$.
17. One root is $x = 2$: $3x^2 - 5x + 2h = 0$.

HINT. — Substitute $x = 2$ in the equation.

18. One root is $x = -3$: $3hx^2 - 5x + 3 = 0$.

19. One root is $x = 0$: $x^2 - 5x + h^2 - 4 = 0$.

20. One root exceeds the other by 5: $x^2 - x - 2h = 0$.

Solution. — 1. Given condition: $r - s = 5$. (1)

2. Sum of roots: $r + s = 1$. (2)

3. Product of roots: $rs = -2h$. (3)

4. Solve (1) and (2) for r and s: $r = 3$ and $s = -2$.

5. Substitute in (3): $3 \cdot (-2) = -2h$; $h = 3$.

21. One root exceeds the other by 2: $4x^2 + 4x + 3h = 0$.

22. One root is the negative of the other: $x^2 - hx + x - 5 + 2h = 0$.

23. One root is four times the other: $x^2 + hx + 16 = 0$.

24. One root is three times the other: $3x^2 + 4 - hx = 0$.

96. Factored form of $(ax^2 + bx + c)$. — *If r and s are the roots of $(ax^2 + bx + c) = 0$, then*

$$ax^2 + bx + c = a(x - r)(x - s). \quad (1)$$

Proof. — **1.** $ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$.

2. From (1) and (2), Section 95, $\frac{b}{a} = -(r + s)$; $\frac{c}{a} = rs$.

Hence, $(ax^2 + bx + c) = a[x^2 - (r + s)x + rs] = a(x - r)(x - s)$.

97. Formation of a quadratic equation with given roots.

Example. — Form a quadratic equation whose roots are $x = -3$ and $x = 5$.

Solution. — By use of Section 96, it follows that the equation is of the form $a(x + 3)(x - 5) = 0$. A simple equation is obtained by placing $a = 1$:

$(x + 3)(x - 5) = 0$, or $x^2 - 2x - 15 = 0$.

EXERCISE 56

Form a quadratic equation with the given numbers as roots:

1. 5, 3.

2. -3, -7.

3. $-\frac{3}{4}$, $\frac{2}{3}$.

4. .1, .5.

5. a, b.

6. $2\sqrt{3}$, $-2\sqrt{3}$.

7. $(1+\sqrt{2}),\ (1-\sqrt{2})$.

8. $(2+3\sqrt{3}),(2-3\sqrt{3})$.

9. $(2+5i),(2-5i)$.

10. $(4-3i),(4+3i)$.

11. $(3-\sqrt{-5}),(3+\sqrt{-5})$.

12. $\frac{2-\sqrt{2}}{3},\ \frac{2+\sqrt{2}}{3}$.

13. $\frac{2-i\sqrt{3}}{2},\ \frac{2+i\sqrt{3}}{2}$.

14. $\frac{c+di}{a},\ \frac{c-di}{a}$.

98. Equations in quadratic form.

Example 1. — Solve for x: $\quad x^4 - 5x^2 + 6 = 0$.

Solution. — 1. Let $y = x^2$; then $y^2 = x^4$, and the given equation becomes $\quad y^2 - 5y + 6 = 0$.

2. Solve for y: $(y-3)(y-2) = 0$; $y = 2$ and $y = 3$.

3. Since $y = x^2$, hence, when $y = 2$, $x^2 = 2$; $x = \pm\sqrt{2} = \pm 1.414$;
when $y = 3$, $x^2 = 3$; $x = \pm\sqrt{3} = \pm 1.732$.

The equation in Example 1 is said to be in the **quadratic form** in x^2.

Example 2. — Solve: $\quad 2x^{-4} - x^{-2} - 3 = 0$.

Solution. — 1. Let $y = x^{-2}$; $y^2 = x^{-4}$, and the given equation becomes $\quad 2y^2 - y - 3 = 0$.

2. Solving for y, we obtain $y = -1$ and $y = \frac{3}{2}$.

3. Since $y = x^{-2}$, hence, when $y = -1$, then $x^{-2} = -1$. Hence,

$$\frac{1}{x^2} = -1, \text{ or } x^2 = -1;\ x = \pm\sqrt{-1} = \pm i.$$

3. When $y = \frac{3}{2}$, $x^{-2} = \frac{3}{2}$; $\frac{1}{x^2} = \frac{3}{2}$; $x^2 = \frac{2}{3}$. Hence,

$$x = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3} = \pm .816.$$

NOTE. — In solving an equation of the form $x^k = A$, where k is a positive integer greater than 2, we agree, for the present, that we desire only *real* solutions, unless otherwise specified. The real solutions, if any, of $x^k = A$ are the real kth roots of A. Thus, $x^4 = -8$ has no real solutions; $x^6 = 64$ has two real solutions, $x = \pm\sqrt[6]{64} = \pm 2$.

EXERCISE 57

Solve by reducing to a quadratic in some new variable:

1. $x^4 - 3x^2 + 2 = 0$.
2. $2x^4 - 11x^2 + 5 = 0$.
3. $4x^{-4} - 9x^{-2} + 2 = 0$.
4. $x^6 - 6x^3 = 16$.
5. $x^4 - 16 = 0$.
6. $y^{-4} - 20y^{-2} + 64 = 0$.
7. $16y^8 - 17y^4 = -1$.
8. $x^{10} - 30x^5 - 64 = 0$.
9. $(x^2 + 6x)^2 - 2(x^2 + 6x) = 35$.
10. $x^{-6} + 19x^{-3} = 216$.
11. $(x^2 - x)^2 - 8(x^2 - x) = -12$.
12. $(x^2 + 3x)^2 - 3x^2 - 9x = 4$.
13. $2w^4 - 7w^2 - 15 = 0$.
14. $3x^6 - 4x^3 - 15 = 0$.
15. $\left(x - \frac{2}{x}\right)^2 - 4\left(x - \frac{2}{x}\right) - 5 = 0$.
16. $\frac{2}{y^2} - \frac{5}{y} + 3 = 0$.
17. $\frac{8}{(x^2 + 3x)^2} + \frac{2}{x^2 + 3x} - 1 = 0$.
18. $x^4 + 2x^3 + x^2 - 8(x^2 + x) = -12$.
19. $\frac{2 + y^2}{2y - 5} - \frac{2y - 5}{2 + y^2} = \frac{35}{6}$.

99. An irrational equation is one in which the variables occur under radical signs, or in expressions with fractional exponents.

NOTE. — Let $M = N$ represent any equation. On squaring both members, we obtain $M^2 = N^2$, which is satisfied if $M = N$ or if $M = -N$. Hence, the solutions of $M^2 = N^2$ consist of all roots of $M = N$ together with all roots of $M = -N$. Therefore, if both members of an equation are squared,* the roots of the new equation consist of the roots of the original equation with, perhaps, other roots which the original equation did not possess.

Thus, $x = 5$ is the only root of the equation $x - 3 = 2$. (1)
Square both members of (1): $(x - 3)^2 = 4$. (2)
Solve (2) for x: $x - 3 = \pm 2$; $x = 5$ and $x = 1$.
Hence, (2) has the root $x = 1$, besides the root $x = 5$ of (1).

* The statement is also true if the word *squared* is changed to "*raised to any positive integral power.*"

Recall that, if N is positive, $\sqrt{N}$ represents the **positive** square root of N.

Example 1. — Solve for x in equations (a) and (b):

$(a)\ 2x - 2 = \sqrt{2x^2 + 4}$.	$(b).\ 2x - 2 = -\sqrt{2x^2 + 4}$.
Solution. — 1. Square both sides:	*Solution.* — 1. Square both sides:
2. $4x^2 - 8x + 4 = 2x^2 + 4$.	2. $4x^2 - 8x + 4 = 2x^2 + 4$.
3. $2x^2 - 8x = 0$; $2x(x-4) = 0$.	3. $2x^2 - 8x = 0$; $2x(x-4) = 0$.
4. $x = 0$ and $x = 4$.	4. $x = 0$ and $x = 4$.
Test. — Substitute $x = 0$ in (a):	*Test.* — Substitute $x = 0$ in (b):
Does $(0 - 2) = \sqrt{4}$? Or, does $-2 = 2$? No.	Does $(0 - 2) = -\sqrt{4}$? Yes.
Substitute $x = 4$ in (a):	Substitute $x = 4$ in (b):
Does $(8 - 2) = \sqrt{36}$? Yes.	Does $(8 - 2) = -\sqrt{36}$? Or, does $6 = -6$? No.
$x = 0$ is *not*, and $x = 4$ *is* a root.	$x = 4$ is *not*, and $x = 0$ *is* a root.

Discussion of Example 1. — If the solution of an equation produces values of the variable which are not roots, we call these values **extraneous** roots. The extraneous root $x = 0$ was introduced in solving (a). In solving either (a) or (b) the test of the values found in Step 4 was necessary because squaring both members of an equation may introduce extraneous roots. The necessity for the test is also shown by the fact that, although (a) and (b) are different equations, all distinction between them is lost after squaring.

Example 2. — Solve: $\sqrt{x - 2} - \sqrt{2x + 5} = 3$.

Solution. — 1. $\sqrt{x - 2} = 3 + \sqrt{2x + 5}$.

2. Square both members: $x - 2 = 9 + 6\sqrt{2x + 5} + 2x + 5$.

3. Simplify: $-x - 16 = 6\sqrt{2x + 5}$.

4. Square both members: $x^2 + 32x + 256 = 36(2x + 5)$,
$x^2 + 32x + 256 = 72x + 180$.

5. $x^2 - 40x + 76 = 0$; $(x - 38)(x - 2) = 0$.

6. Possible roots of the original equation are $x = 38$ and $x = 2$.

Test. — Substitute $x = 2$ in the original equation:

Does $\sqrt{2 - 2} - \sqrt{4 + 5} = 3$? Does $-3 = 3$? No.

Substitute $x = 38$ in the original equation:

Does $\sqrt{38 - 2} - \sqrt{76 + 5} = 3$? Does $\sqrt{36} - \sqrt{81} = 3$? No.

Hence, the original equation has no roots.

Method for solving an equation involving radicals:

1. *Transpose the most complicated radical to one member and all other terms to the other member.*

2. *If the complicated radical is a square root, square both members; if a cube root, cube both members; etc.*

3. *Repeat Steps* 1 *and* 2 *until an equation is obtained free of radicals involving the unknown; solve the resulting equation.*

4. *Test each value, found in Step* 3, *by substitution in the original equation, to determine which values are roots.*

To solve an irrational equation involving fractional powers, first express each such power as a radical.

EXERCISE 58

Solve:

1. $\sqrt{x+2}=3$.
2. $\sqrt[3]{x+2}=3$.
3. $\sqrt{3x+2}=-3$.
4. $\sqrt[4]{y-1}=2$.
5. $y^{\frac{1}{4}}=2$.
6. $(x+1)^{\frac{1}{2}}=5$.
7. $2x=\sqrt{2}$.
8. $\sqrt[3]{x^2}=4$.
9. $3\sqrt{y}=2+y$.
10. $\sqrt[3]{x^2-2x}-2=0$.
11. $\sqrt{x^2+x}=-3\sqrt{x+1}$.
12. $\sqrt[4]{x^2-24x}-3=0$.
13. $y^2-y\sqrt{3}=0$.
14. $\sqrt{x+4}=\sqrt{x}+1$.
15. $3\sqrt{z+1}=\sqrt{3z}+3$.
16. $\sqrt{x-\sqrt{x+3}}=3$.
17. $\sqrt{2y-1}+\sqrt{2y+6}=7$.
18. $\sqrt{2x-5}-\sqrt{x-2}=2$.
19. $\sqrt{3-x}-\sqrt{2+x}=3$.
20. $2\sqrt{3z-2}+3\sqrt{z-2}=2$.
21. $(x+7)^{\frac{1}{2}}=2-\sqrt{x-5}$.
22. $(2x+3)^{\frac{1}{2}}-(x-2)^{\frac{1}{2}}+2=0$.
23. $\sqrt{\dfrac{x-3}{x+3}}=2$.
24. $\dfrac{\sqrt{y}+3}{\sqrt{2y+9}}=1$.
25. $\sqrt{2-3y}+\sqrt{3+y}=\sqrt{1+4y}$.
26. $x^{\frac{1}{2}}=(x+3)^{\frac{1}{2}}+(x+8)^{\frac{1}{2}}$.

SUPPLEMENTARY EXERCISE 59

NOTE. — Sometimes, by mere inspection we can decide when an equation of the form $x^{\frac{m}{n}} = A$ has no real roots. Thus, $x^{\frac{3}{4}} = -2$ has no real roots because $x^{\frac{3}{4}}$ represents the principal fourth root of x^3, and a principal fourth root of x^3 is either positive or imaginary if x is real.

Example. — Solve: $x^{\frac{3}{4}} = 8$.

Solution. — 1. Raise to 4th power: $x^3 = 8^4 = (2^3)^4 = 2^{12}$.

2. Hence, $x = \sqrt[3]{2^{12}} = 2^4 = 16$.

Check. — Does $(2^4)^{\frac{3}{4}} = 8$? Does $2^3 = 8$? Yes.

Find all real roots; if there are none, recognize this by inspection:

1. $y^{\frac{3}{2}} = 8$. **3.** $z^{\frac{3}{5}} = -5$. **5.** $x^{\frac{2}{5}} = 9$. **7.** $x^{\frac{3}{5}} = -8$.

2. $y^{\frac{3}{4}} = -8$. **4.** $x^{\frac{2}{3}} = 16$. **6.** $x^{\frac{2}{3}} = -16$. **8.** $x^{-\frac{3}{4}} = 8$.

Solve by reducing to a quadratic in some new letter:

9. $(x^2 + 5) + 2(x^2 + 5)^{\frac{1}{2}} = 15$. **10.** $5w - 9\sqrt{w} = 2$.

HINT for Problem 9. — Let $y = (x^2 + 5)^{\frac{1}{2}} = \sqrt{x^2 + 5}$.

11. $2x - 15\sqrt{x} - 27 = 0$. **12.** $2x^{\frac{2}{3}} - 3x^{\frac{1}{3}} - 2 = 0$.

13. $x^2 - 2x - 6 - \sqrt{x^2 - 2x - 6} = 6$.

14. $2x^2 - 5x - \sqrt{2x^2 - 5x + 1} = 1$.

15. $3x^{\frac{1}{2}} - 14x^{\frac{1}{4}} - 5 = 0$. **17.** $6x^{-\frac{1}{2}} - x^{-\frac{1}{4}} - 1 = 0$.

16. $3x^3 - 26x^{\frac{3}{2}} - 9 = 0$. **18.** $2x^{-3} - 15x^{-\frac{3}{2}} - 8 = 0$.

Solve for x:

19. $\sqrt{10x + 6a} - \sqrt{3x} = \sqrt{x + 6a}$.

20. $\sqrt{3 - x} - \sqrt{7 + 3x} = \sqrt{1 + x}$.

21. $\sqrt{x + a} = \sqrt{3x} - \sqrt{a}$.

22. $\sqrt{x} + \sqrt{3x + 4b} = \sqrt{8x + 4b}$.

23. Solve $v = \sqrt{2gs}$, (*a*) for s; (*b*) for g.

24. Solve $t = \pi\sqrt{\frac{l}{g}}$, (*a*) for l; (*b*) for g.

25. Solve for h: $l = \pi r\sqrt{r^2 + h^2}$.

26. Solve for h: $r = .38\sqrt{\pi d\sqrt{h}}$.

Prove the theorems stated in Problems 27, 28, *and* 29:

27. (*A*) *If* $(ax^2 + bx + c) = 0$ *has one root zero, then* $c = 0$. (*B*) *If* $c = 0$, *then one root of* $(ax^2 + bx + c) = 0$ *is* $x = 0$.

28. (*A*) *If both roots of* $(ax^2 + bx + c) = 0$ *are zero, then* $b = 0$ *and* $c = 0$. (*B*) *If* $b = 0$ *and* $c = 0$, *then the equation* $(ax^2 + bx + c) = 0$ *has both roots zero.*

HINT for (*A*). — Use the formulas 1 and 2, Section 95.

29. *If the roots of* $(ax^2 + bx + c) = 0$ *are numerically equal, but opposite in sign, then* $b = 0$.

HINT. — If r and s are numerically equal but opposite in sign, then

$$r = -s.$$

In Problems 30 *to* 32, *by use of the results in Problem* 28, *find the values of the constants* h *and* k *if both roots of the equation are zero:*

30. $3x^2 - 4x + 3hx + kx + 5h - k - 4 = 0$.

HINT. — 1. Collect terms:

$$3x^2 + x(-4 + 3h + k) + (5h - k - 4) = 0.$$

2. By Problem 28, Part *A*, $\begin{cases} -4 + 3h + k = 0, & (1) \\ 5h - k - 4 = 0. & (2) \end{cases}$

31. $2x^2 + x(h - 2k - 3) + 2h + 2k = 0$.

32. $5x^2 - 2hx + 3kx + h - x + 3k - 1 = 0$.

33. Find the value of h if the roots of the equation are numerically equal but opposite in sign:

(*a*) $3hx^2 - 5x + 20h^2x - 3 = 0$.

(*b*) $x^2 - 2x + 5hx + 3h^2x - 4 = 0$.

Factor by first solving a quadratic equation:

34. $6x^2 - 23x + 20$.

Solution. — 1. Solve $(6x^2 - 23x + 20) = 0$; $x = \frac{5}{2}$ and $x = \frac{4}{3}$.

2. By equation 1, Section 96,

$$6x^2 - 23x + 20 = 6(x - \tfrac{5}{2})(x - \tfrac{4}{3}) = (2x - 5)(3x - 4).$$

35. $12x^2 + 11x - 36$.

36. $40x^2 - 23x - 30$.

37. $27y^2 - 6y - 16$.

38. $15y^2 - 2y - 24$.

39. $27x^2 - 21ax - 40a^2$.

40. $12x^2 + xw - 20w^2$.

NOTE. — If a, b, and c are rational numbers, $(ax^2 + bx + c)$ can be factored into linear factors involving only rational numbers, when and only when $(ax^2 + bx + c) = 0$ has *rational roots.*

Without actually factoring, and without solving any equation, determine whether the expression has rational linear factors:

41. $5x^2 - 3x + 7$.

42. $2x^2 - 8x + 8$.

43. $12y^2 + 8y - 45$.

44. $21x^2 + 23x + 6$.

45. $11w^2 + 12wy - 25y^2$.

46. $6x^2 - 25xy + 25y^2$.

Solve by use of factoring:

Example. — $27x^3 = 8$.

Solution. — 1. By formula 1, Section 16, $27x^3 - 8 = 0$ becomes $(3x - 2)(9x^2 + 6x + 4) = 0$; hence, $3x - 2 = 0$, or $9x^2 + 6x + 4 = 0$.

2. If $3x - 2 = 0$, $x = \frac{2}{3}$; if $9x^2 + 6x + 4 = 0$, $x = \frac{-1 \pm i\sqrt{3}}{3}$.

47. $x^3 - 27 = 0$.

48. $x^3 = 1$.

49. $x^4 = 64$.

50. $x^4 = 16$.

51. $81y^4 - 16 = 0$.

52. $8y^3 + 125 = 0$.

53. $125x^3 = -1$.

54. $A^3 = 64$.

55. $A^4 = \frac{1}{16}$.

56. Find the three cube roots: (*a*) of 125; (*b*) of $\frac{1}{8}$; (*c*) of -8.

HINT. — In Problem 54, the solutions are the three cube roots of 64.

57. Find the four fourth roots of 81; of 625.

CHAPTER XII

SYSTEMS OF EQUATIONS INVOLVING QUADRATICS

100. Graphs of quadratic equations in two variables.

Example 1. — Graph the equation:

$$x^2 + y^2 = 25.$$

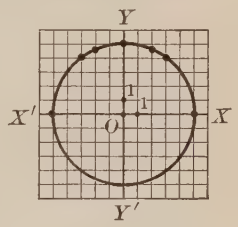


Fig. 9.

Solution. — 1. *Solve for* y: $y^2 = 25 - x^2$, and hence

(*a*) $y = +\sqrt{25 - x^2}$, or (*b*) $y = -\sqrt{25 - x^2}$.

2. *The tables of values of* x *and* y. — Notice that $(25 - x^2) = 0$ if $x = \pm 5$, and that $\sqrt{25 - x^2}$ is imaginary if $x > 5$ or $x < -5$. Thus, if $x = -6$, $\sqrt{25 - x^2} = \sqrt{-11}$. Hence, in the following tables, we assign values to x only from $x = -5$ to $x = +5$.

(*a*)	IF $x =$	-5	-3	-2	0	2	3	5
$y = \sqrt{25 - x^2}$	THEN $y =$	0	4	4.5	5	4.5	4	0
(*b*)	IF $x =$	-5	-3	-2	0	2	3	5
$y = -\sqrt{25 - x^2}$	THEN $y =$	0	-4	-4.5	-5	-4.5	-4	0

3. In Figure 9, the *upper half* of the circle is the graph of (*a*), and the *lower half* is the graph of (*b*); the *whole* circle is the graph of $x^2 + y^2 = 25$.

101. Graphical solution of systems involving quadratics.

Example 1. — Solve graphically:

$$\begin{cases} x^2 + y^2 = 25, & (1) \\ x + y - 1 = 0. & (2) \end{cases}$$

Solution. — 1. In Figure 10, the circle is the graph of (1) and the straight line is the graph of (2). Any point on the circle has coördinates which satisfy (1) and any point on the straight line has coördinates

which satisfy (2). Hence, *both* equations are satisfied by the coördinates of the following points, *which lie on both graphs:*

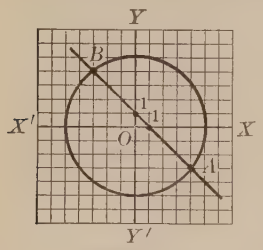


FIG. 10.

A: $(x = 4, y = -3)$. B: $(x = -3, y = 4)$.

These pairs of values of x and y are the two solutions of the system, and they can be checked by substitution in (1) and (2).

NOTE. — Only real solutions can be found graphically and, usually, solutions can be read only approximately, from a graph.

Example 2. — Solve graphically: $\begin{cases} 16x^2 + 49y^2 = 784, & (3) \\ 9x^2 - 4y^2 = 36. & (4) \end{cases}$

Solution. — 1. In Figure 11, the graph of (3) is the oval curve, which is called an ellipse. This graph was obtained by the method of Example 1, Section 100.

2. *To graph* (4). — Solve for x: $x = \pm \frac{2}{3}\sqrt{9 + y^2}$.

We assign values to y and compute the corresponding values of x.

(a)	IF $y =$	-10	-8	-3	0	3	8	10
$x = \frac{2}{3}\sqrt{9 + y^2}$	THEN $x =$	7.0	5.7	2.8	2	2.8	5.7	7.0
(b)	IF $y =$	-10	-8	-3	0	3	8	10
$x = -\frac{2}{3}\sqrt{9 + y^2}$	THEN $x =$	-7.0	-5.7	-2.8	-2	-2.8	-5.7	-7.0

In Figure 11, the graph of (a) is the open curve at the right of the y-axis, and the graph of (b) is the open curve at the left. *These two open curves, together,* are called an **hyperbola**, and it is the graph of (4). Each piece of the hyperbola is called a **branch** of it.

3. The ellipse and the hyperbola intersect in the following points:

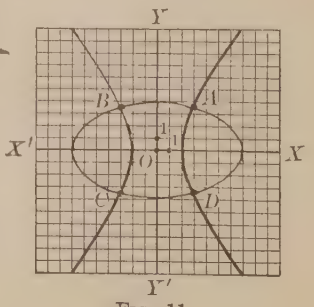


FIG. 11.

A: $(x = 3,\ y = 3.5)$. C: $(x = -3,\ y = -3.5)$.

B: $(x = -3,\ y = 3.5)$. D: $(x = 3,\ y = -3.5)$.

These pairs of values of x and y are the solutions of the system.

NOTE. — In Step 2 of the solution of Example 2, we solved for x instead of for y because we thus obtain a radicand, $(9 + y^2)$, which is positive for all values of y. The solution for y in terms of x would bring in a less convenient radicand.

Facts concerning graphs, proved in advanced mathematics:

1. *If c is positive, the graph of* $x^2 + y^2 = c$ *is a* **circle** *whose radius is* $\sqrt{c}$ *and center is the origin,* provided that the same unit is used on the scales of the x-axis and y-axis.

2. *If a, b, and c have the same sign, the graph of* $ax^2 + by^2 = c$ *is an* **ellipse**, *with the center at the origin.*

3. *If a and b have opposite signs, the graph of* $ax^2 + by^2 = c$ *is an* **hyperbola.**

Thus, the graph of $3x^2 + 7y^2 = 8$ is an ellipse, and of $5x^2 - y^2 = 7$ is an hyperbola.

4. *The graph of any quadratic equation in x and y is* **either an ellipse, an hyperbola, a parabola,** *or* **a pair of straight lines.**

EXERCISE 60

Solve graphically:

1. $\begin{cases} 2x - y = 3, \\ x^2 + y^2 = 9. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 25, \\ x + 2y = 4. \end{cases}$

3. $\begin{cases} 4x^2 + 9y^2 = 36, \\ y = x^2. \end{cases}$

4. $\begin{cases} x^2 + y^2 = 36, \\ 4y = x^2 - 20. \end{cases}$

5. $\begin{cases} x^2 - y^2 = 16, \\ y - x = 2. \end{cases}$

6. $\begin{cases} 25y^2 - 4x^2 = 100, \\ 4x^2 + y^2 = 36. \end{cases}$

NOTE. — In Problem 7, the graph of the first equation is a pair of straight lines; to obtain them, solve for x in terms of y. In Problem 8, the graph of $xy = 12$ is an hyperbola; one branch lies above the x-axis, to the right of the y-axis.

7. $\begin{cases} 2x^2 - xy - 6y^2 = 0, \\ x^2 + y^2 = 25. \end{cases}$

8. $\begin{cases} y = 2x - 2, \\ xy = 12. \end{cases}$

9. $\begin{cases} 4x^2 - 9y^2 = 0, \\ 4x^2 + 9y^2 = 36. \end{cases}$

10. $\begin{cases} y = 2x^2 - 4x - 7, \\ xy = 15. \end{cases}$

102. A system consisting of one linear and one quadratic equation can be solved by elimination by substitution.

Example 1. — Solve: $\begin{cases} 4x^2 - 6xy + 9y^2 = 63, & (1) \\ 2x - 3y = -3. & (2) \end{cases}$

Solution. — 1. Solve (2) for x: $x = \dfrac{3y-3}{2}$. (3)

2. Substitute (3) in (1) and solve for y:

$$4\left(\frac{3y-3}{2}\right)^2 - 6y\left(\frac{3y-3}{2}\right) + 9y^2 = 63. \quad (4)$$

$y^2 - y - 6 = 0$; $(y-3)(y+2) = 0$; $y = 3$ and $y = -2$.

3. In equation 3, if $y = 3$, then $x = 3$; if $y = -2$, then $x = -\frac{9}{2}$.

4. The solutions are $\boxed{x = 3,\ y = 3}$ and $\boxed{x = -\tfrac{9}{2},\ y = -2}$.

These solutions check when substituted in (1) and (2).

NOTE. — A consistent system of one linear and one quadratic equation may have (*a*) two different real solutions, or (*b*) two real solutions which are the same, or (*c*) two imaginary solutions. These three possibilities correspond, respectively, to the following geometrical possibilities: the straight line, which is the graph of the linear equation, (*a*) may cut the graph of the quadratic in two points, or (*b*) may be tangent to, or (*c*) may not touch the graph of the quadratic. The three possibilities are illustrated by Problems 1, 2, and 3 in Exercise 61.

EXERCISE 61

Solve each system, (a) algebraically and (b) graphically:

1. $\begin{cases} x^2+y^2=25, \\ x=15-3y. \end{cases}$ **2.** $\begin{cases} x^2+y^2=25, \\ 4y+3x=25. \end{cases}$ **3.** $\begin{cases} x^2+y^2=25, \\ 4y+3x=50. \end{cases}$

Solve algebraically:

4. $\begin{cases} x^2 - xy + y^2 = 7, \\ x + y = 4. \end{cases}$

5. $\begin{cases} c^2 + d^2 = 29, \\ c + d + 3 = 0. \end{cases}$

6. $\begin{cases} 4x^2 + y^2 = 25, \\ 2x - 7 + y = 0. \end{cases}$

7. $\begin{cases} 3x^2 + 3y^2 - 10xy = 0, \\ x - 1 = y. \end{cases}$

8. $\begin{cases} x + y + 1 = 0, \\ x^2 + 2y^2 - 12y - 4x = 14. \end{cases}$

9. $\begin{cases} 4x^2 - 25y^2 = 100, \\ 5y - x + 5 = 0. \end{cases}$

10. $\begin{cases} 2x - 3y = 1, \\ 3x^2 - 4y^2 - 5xy + 2 = 0. \end{cases}$

11. $\begin{cases} y + 6x = 3x^2 + 8, \\ 2x - 2y = 5. \end{cases}$

12. $\begin{cases} 15uv = 2u + 6v, \\ 6v + 2u = 5. \end{cases}$

13. $\begin{cases} 3x^2 = 5xy + 27, \\ x + y = 3. \end{cases}$

103. When both equations of a system are quadratic, the system usually has **four different solutions**, some or all of which may be imaginary numbers. The student should recall his graphical solutions of systems of this type where four solutions were obtained.

NOTE 1. — The fact stated in the preceding paragraph is a special case of the following theorem which is proved in a later course in algebra: *A system of two integral rational equations in x and y in which one equation is of degree m and the other is of degree n in x and y, usually has mn solutions.* Thus, a system consisting of an equation of the third degree and a quadratic usually has 3×2 or 6 solutions.

NOTE 2. — Usually, the solution of two simultaneous quadratics involves the solution of a fourth degree equation in one variable. At this stage of the course, the student is able to solve only very simple fourth degree equations and hence he is not prepared to consider the solution of all systems of simultaneous quadratics. Therefore, in this chapter we consider only special types of systems, where the solutions can be obtained by elementary means.

104. When both equations have the form $Ax^2 + By^2 = C$, the system is linear in x^2 and y^2 and can be solved for x^2 and y^2 by the methods applicable to systems of linear equations.

Example 1. — Solve: $\begin{cases} x^2 + y^2 = 25, & (1) \\ x^2 + 2y^2 = 34. & (2) \end{cases}$

Solution. — 1. Multiply by 2 in (1): $2x^2 + 2y^2 = 50.$ (3)

2. Subtract, (3) − (2): $x^2 = 16;\ x = \pm 4.$ (4)

3. Solving for y^2, similarly, $y^2 = 9;\ y = \pm 3.$ (5)

4. The solutions consist of all possible pairs which can be formed from $x = \pm 4$ and $y = \pm 3$, because if x is either 4 or -4, and if y is either 3 or -3, the system is satisfied. The solutions are as follows:

$\boxed{x = 4, y = 3}$, $\boxed{x = -4, y = 3}$, $\boxed{x = 4, y = -3}$, $\boxed{x = -4, y = -3}$.

EXERCISE 62

Solve each system, (a) algebraically and (b) graphically:

1. $\begin{cases} x^2 + y^2 = 49, \\ x^2 + 4\,y^2 = 64. \end{cases}$ 2. $\begin{cases} x^2 + y^2 = 4, \\ 9\,x^2 - 4\,y^2 = 36. \end{cases}$ 3. $\begin{cases} x^2 + y^2 = 10, \\ x^2 + 4\,y^2 = 64. \end{cases}$

Solve algebraically:

4. $\begin{cases} 2\,x^2 + y^2 = 7, \\ x^2 - 2\,y^2 + 4 = 0. \end{cases}$ 6. $\begin{cases} 6\,x^2 + 5\,y^2 = 58, \\ x^2 - 2\,y^2 = 4. \end{cases}$

5. $\begin{cases} 4\,u^2 + v^2 = 13, \\ 8\,u^2 - 3\,v^2 = 6. \end{cases}$ 7. $\begin{cases} 4\,x^2 - 6\,y^2 + 1 = 0, \\ 4\,x^2 + 2\,y^2 = 5. \end{cases}$

105. When all terms involving the variables are of the second degree, a system can be solved by the method of Example 1, below.

Example 1. — Solve: $\begin{cases} x^2 + xy + y^2 = 63, & (1) \\ x^2 - y^2 = -27. & (2) \end{cases}$

Solution. — 1. Let $y = wx$. Then, to find the values of x and y, we shall first find the values of w and x which satisfy the following system:

when $y = wx$ in (1), $x^2 + wx^2 + w^2x^2 = 63$, (3)

when $y = wx$ in (2), $x^2 - w^2x^2 = -27$. (4)

2. Solve (3) for x^2: $$x^2 = \frac{63}{1 + w + w^2}. \quad (5)$$

3. Solve (4) for x^2: $$x^2 = \frac{27}{w^2 - 1}. \quad (6)$$

4. Equate expressions for x^2: $$\frac{27}{w^2 - 1} = \frac{63}{1 + w + w^2}. \quad (7)$$

5. Simplify (7) and solve: $4\,w^2 - 3\,w - 10 = 0$; hence,

$$w = 2, \text{ and } w = -\tfrac{5}{4}. \quad (8)$$

6. Substitute $w = 2$ in (6): $x^2 = 9$; $x = \pm 3$. Since $y = wx$, hence, if $x = 3$, then $y = 6$; if $x = -3$, then $y = -6$.

7. Substitute $w = -\frac{5}{4}$ in (6): $x^2 = 48$; $x = \pm 4\sqrt{3}$. Since $y = wx$, hence, if $x = 4\sqrt{3}$, $y = -\frac{5}{4}(4\sqrt{3}) = -5\sqrt{3}$; if $x = -4\sqrt{3}$, $y = 5\sqrt{3}$.

Summary:

CORRESPONDING TO	$w = 2$	$w = -\frac{5}{4}$
THE SOLUTIONS ARE	$x = 3, y = 6,$	$x = 4\sqrt{3}, y = -5\sqrt{3}.$
	$x = -3, y = -6.$	$x = -4\sqrt{3}, y = 5\sqrt{3}.$

EXERCISE 63

Solve algebraically:

1. $\begin{cases} x^2 + 6xy = 28, \\ xy + 8y^2 = 4. \end{cases}$

2. $\begin{cases} 2x^2 - 5xy + 3y^2 = 20, \\ 2x^2 - xy = 8. \end{cases}$

3. $\begin{cases} 4x^2 + 3y^2 = 28, \\ 2x^2 - xy + 2y^2 = 20. \end{cases}$

4. $\begin{cases} h^2 + 20r^2 - rh = 110, \\ 5r^2 - 4rh + h^2 = 65. \end{cases}$

5. $\begin{cases} y^2 + xy = 21, \\ x^2 - xy = 4. \end{cases}$

6. $\begin{cases} u^2 + uz + 10z^2 = 22, \\ uz - 2z^2 + 6 = 0. \end{cases}$

7. $\begin{cases} x^2 - 3xy = 1, \\ 2x^2 + 18y^2 = 5. \end{cases}$

8. $\begin{cases} x^2 - 2xy = 5, \\ x^2 + y^2 = 29. \end{cases}$

106. Reduction to simple systems.

Example 1. — Solve: $\begin{cases} x^2 + y^2 = 14, & (1) \\ x^2 - 3xy + 2y^2 = 0. & (2) \end{cases}$

Solution. — 1. Factor (2): $(x - 2y)(x - y) = 0.$ (3)

2. Hence, (2) is satisfied if either $(x - 2y) = 0$ or $(x - y) = 0.$

3. Hence, (1) and (2) are satisfied if and only if x and y satisfy one of the following systems: $\begin{cases} x^2 + y^2 = 14, \\ x - y = 0. \end{cases}$ $\begin{cases} x^2 + y^2 = 14, \\ x - 2y = 0. \end{cases}$ (4)

4. It is left to the student to solve systems (4), each of which has two solutions, and thus to obtain the four solutions of the given system.

Two systems are **equivalent** if they have the same solutions. **A system in which all terms involving the variables are of the second degree*** can sometimes be reduced to an equivalent system where one equation can be factored.

*In Section 105, one method has already been given for solving this type.

Example 2. — Solve: $\begin{cases} x^2 + 3xy = 28, & (5) \\ xy + 4y^2 = 8. & (6) \end{cases}$

Solution. — 1. *Eliminate the constants:*

Multiply (5) by 2: $2x^2 + 6xy = 56.$ (7)

Multiply (6) by 7: $7xy + 28y^2 = 56$ (8)

Subtract (7) − (8): $2x^2 - xy - 28y^2 = 0$; or, (9)

$$(2x + 7y)(x - 4y) = 0.$$

2. To solve the given system, we solve each of the following systems:

$$\begin{cases} xy + 4y^2 = 8, \\ 2x + 7y = 0. \end{cases} \qquad \begin{cases} xy + 4y^2 = 8, \\ x - 4y = 0. \end{cases} \tag{10}$$

Instead of using (6) in (10), we could equally well have used (5). It is left to the student to solve systems (10), each of which has two solutions, and thus to obtain the four solutions of [(5), (6)].

In Example 1, *one quadratic equation was changed to two equivalent linear equations.* This same feature appears in the next example.

Example 3. — Solve: $\begin{cases} x^2 + xy + y^2 = 20, & (11) \\ xy = 5. & (12) \end{cases}$

Solution. — 1. (11) + (12): $x^2 + 2xy + y^2 = 25.$ (13)

2. From (13), $(x + y)^2 = 25$; $x + y = 5$, or $x + y = -5.$ (14)

3. To solve [(11), (12)], we solve each of the following systems:

$$\begin{cases} x + y = 5, \\ xy = 5. \end{cases} \qquad \begin{cases} x + y = -5, \\ xy = 5. \end{cases} \tag{15}$$

4. The student may complete the solution.

Hereafter, **to solve** will mean **to solve algebraically.**

EXERCISE 64

Solve by reducing to one or more systems involving a linear equation:

1. $\begin{cases} 2x^2 - 5xy - 3y^2 = 0, \\ 2x^2 - 3xy = 2. \end{cases}$

2. $\begin{cases} 3x^2 - 5xy - 2y^2 = 0, \\ x^2 - xy = 4. \end{cases}$

3. $\begin{cases} xy + 4y^2 = 8, \\ x^2 + 2xy - 4y^2 = 20. \end{cases}$

4. $\begin{cases} 3u^2 + uv - v^2 = 9, \\ v^2 - u^2 = 27. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 50, \\ xy = 7. \end{cases}$

6. $\begin{cases} 2x^2 - 7xy + 6y^2 = 0, \\ x^2 + 3y^2 = 7. \end{cases}$

7. $\begin{cases} 3xy - 2y^2 = 0, \\ 2x^2 + xy - 2y^2 = 2. \end{cases}$

8. $\begin{cases} x^2 - 3xy + 2y^2 = 3, \\ x^2 - y^2 + 3xy = 3. \end{cases}$

9. $\begin{cases} x^2 + xy = 36, \\ x^2 + 2y^2 - 3xy = 6. \end{cases}$

10. $\begin{cases} 5xy - x = 6, \\ 2xy + 3y = 7. \end{cases}$

HINT for Problem 5. — Multiply both sides of the second equation by 2, and add to first equation.

Solve by two methods:

11. $\begin{cases} 3xy - 4y^2 = 2, \\ 3x^2 - 9xy + 7y^2 = 1. \end{cases}$

12. $\begin{cases} x^2 + 4y^2 = 13, \\ xy = 3. \end{cases}$

SUPPLEMENTARY PROBLEMS

Solve by first reducing to one or more simpler systems:

13. $\begin{cases} x^3 + y^3 = 27, & (1) \\ x + y = 3. & (2) \end{cases}$

HINT. — 1. Factor (1): $(x + y)(x^2 - xy + y^2) = 27.$ (3)

Divide (3) by (2): $x^2 - xy + y^2 = 9.$ (4)

2. Hence, (x, y) satisfies [(1), (2)] if and only if (x, y) satisfies (2) and (4). Complete the solution by solving the system consisting of (2) and (4).

14. $\begin{cases} 2a + b = 2, \\ 8a^3 + b^3 = 98. \end{cases}$ **15.** $\begin{cases} x^4 - y^4 = 240, \\ x^2 + y^2 = 20. \end{cases}$ **16.** $\begin{cases} x(x - y) = 40, \\ y(x - y) = 20. \end{cases}$

17. $\begin{cases} x^2 + y^2 + 2x + 2y = 8, & (1) \\ 2xy + x + y = -4. & (2) \end{cases}$

HINT. — 1. Let $x = u + v$ and $y = u - v$ in (1) and (2); then, from (1), $2u^2 + 2v^2 + 4u = 8,$ (3)

and, from (2), $2u^2 - 2v^2 + 2u = -4.$ (4)

2. Eliminate v^2; (3) + (4): $4u^2 + 6u - 4 = 0.$ (5)

Solve (5) for u, and then find the corresponding values of v from (3). From each pair of values of (u, v), obtain a solution of [(1), (2)] by use of $x = u + v$ and $y = u - v$.

NOTE. — An equation is symmetrical in x and y if, when x and y are interchanged, the equation is unaltered. In a symmetrical quadratic in x and y, with one member zero, x^2 and y^2 have the same coefficient, and x and y have the same coefficient. The method of the preceding solution applies when each equation is symmetrical.

Solve by the method used in Problem **17**:

18. $\begin{cases} x^2+4x+y^2+4y = 17, \\ xy + 6 = 0. \end{cases}$ **19.** $\begin{cases} x^2 - 3xy + y^2 = 1, \\ 2x^2 - xy + 2y^2 = 17. \end{cases}$

Solve:

20. $\begin{cases} (x + y)(x + 2y)(3x - y) = 0, \\ (2x - y + 1)(x + 3y - 2) = 0. \end{cases}$

21. $\begin{cases} 3x^2 + 3y^2 + 3xy + 2x + 2y = 3, \\ 9x^2 - 9xy + 9y^2 = 7. \end{cases}$

22. $\begin{cases} 27w^3 - v^3 = 105, \\ 3w - v = 15. \end{cases}$ **23.** $\begin{cases} (x + z)^2 - 3(x + z) = 4, \\ x^2 + 2xz + 2z^2 = 17. \end{cases}$

MISCELLANEOUS EXERCISE 65

Solve:

1. $\begin{cases} 4x^2 + y^2 = 25, \\ 2x + y = 7. \end{cases}$ **3.** $\begin{cases} 2x^2 - 3xy = 2, \\ 4x^2 + 9y^2 = 10. \end{cases}$

2. $\begin{cases} x^2 + y^2 - 6y = 0, \\ y + 2x = 0. \end{cases}$ **4.** $\begin{cases} 3u^2 - 5uv + 3v^2 = 15, \\ u^2 - uv = 4. \end{cases}$

5. $\begin{cases} (x - y)(x + 2y - 1) = 0, \\ x^2 - 2xy + 2x + 2y = 3y^2. \end{cases}$

6. $\begin{cases} 2y^2 - xy = 0, \\ y^2 + 2x^2 = 9. \end{cases}$ **7.** $\begin{cases} 4x^2 + 9y^2 = 13, \\ 8x^2 - 27y^2 = 6. \end{cases}$ **8.** $\begin{cases} uv - 4u^2 = 1, \\ v^2 - 3uv = 10. \end{cases}$

Solve, (a) graphically and (b) algebraically:

9. $\begin{cases} x^2 + y^2 = 25, \\ 2y^2 - x^2 = 2. \end{cases}$ **10.** $\begin{cases} 2y^2 + 5x^2 = 10, \\ x - y = 5. \end{cases}$

Solve, without first clearing of fractions:

11. $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{y^2} = 74, \\ \dfrac{1}{x} - \dfrac{1}{y} = 12. \end{cases}$ **12.** $\begin{cases} \dfrac{1}{x} - \dfrac{1}{y} = 11, \\ \dfrac{1}{xy} = -28. \end{cases}$

HINT. — Let $u = \dfrac{1}{x}$ and $v = \dfrac{1}{y}$; find u and v; then find x and y.

13. Find two numbers whose sum is -9 and the sum of whose squares is 101.

14. The product of two numbers is 6. The sum of the squares of the numbers is $\frac{73}{4}$. Find the numbers.

15. The altitude of a certain rectangle is 2 feet longer than the side of a certain square. The perimeter of the rectangle is 7 times the side of the square. The area of the rectangle is 32 square feet more than twice the area of the square. Find the dimensions of the rectangle and of the square.

16. Find two numbers, the sum of whose squares is $\frac{37}{2}$, if twice the first number, minus the second number, equals $\frac{9}{2}$.

17. Find two numbers such that their product increased by the first number equals 8, and their product increased by twice the second number equals 12.

18. Find the dimensions of a rectangle whose area is 48 square feet, if the length of the diagonal is 10 feet.

19. The product of two numbers is 60. If the first number is multiplied by the sum of the numbers, the result is $\frac{265}{4}$. Find the two numbers.

20. Two men, working together, can do a certain piece of work in 30 days. Find the number of days it would take each to do the work alone if the sum of these numbers of days is 125.

21. The sum of the squares of the two digits of a certain positive integral number is 45. If 27 is added to the number, the digits of the result are the digits of the original number in reverse order. Find the original number.

HINT. — If t and u are the digits, the original number is $(10\,t + u)$.

22. The sum of the squares of the two digits of a certain positive integral number is 106. If the number itself is 3 less than seven times the sum of its digits, find the number.

23. In eight hours, two boys are able to row 12 miles downstream, and back to their starting point. Their rate, rowing downstream, is three times the rate of the current. Find the

rate of the current and the rate at which the boys can row in still water.

HINT. — If x and y are the rates, the time going downstream is $12 \div (x + y)$.

24. Two towns on opposite sides of a lake are 28 miles apart, by water. At 6 A.M., from each town a boat starts for the other town, traveling at uniform speed. The boats pass each other at 8 A.M.. One boat arrives at its destination 1 hour and 10 minutes earlier than the other. What is the rate of each boat?

25. A wheel of one automobile makes 96 more revolutions per mile than a wheel of a second automobile. If 20 inches were added to the length of the radius of a wheel of the first automobile, the result would be the diameter of a wheel of the second automobile. Find the diameter of a wheel of each automobile, using $\frac{22}{7}$ as the approximate value of π.

26. A man has two square lots of different size, containing 289 square yards. If the lots were placed side by side so as to form one piece of ground having six sides, the perimeter of the piece would be 76 yards. Find the length of a side of each lot.

SUPPLEMENTARY PROBLEMS

Find the values of the constant k for which the straight line, which is the graph of the linear equation, will be tangent to the graph of the quadratic equation:

27. $\begin{cases} x + y = 1, & (1) \\ x^2 + y^2 = k^2. & (2) \end{cases}$

HINT. — 1. Solve (1) for y: $y = 1 - x$. (3)

2. Substitute (3) in (2) and simplify: $2x^2 - 2x + (1 - k^2) = 0$. (4)

3. If the straight line is tangent, then (4) has equal roots. Find the values of k for which this is true.

28. $\begin{cases} x^2 + y^2 = 16, \\ kx + y = 5. \end{cases}$ **29.** $\begin{cases} x^2 + y^2 = 25, \\ y - 4 = kx - 3k. \end{cases}$ **30.** $\begin{cases} x^2 + y^2 = k^2, \\ x + 2y = 3. \end{cases}$

31. Find a relation between a, b, c and m, given that the system has equal solutions for x and y and given that $a \neq 0$ and $b \neq 0$.

$$\begin{cases} b^2x^2 - a^2y^2 = a^2b^2, \\ y = mx + c. \end{cases}$$

Solve:

32. $\begin{cases} w(1 + .03\,r) = 840, \\ w(1 + .07\,r) = 960. \end{cases}$

33. $\begin{cases} x^2 + y^2 = 16 + 3\,x, \\ x^2 - y^2 = 4. \end{cases}$

34. $\begin{cases} u^2 + 2\,uv + v^2 - 3\,u - 3\,v = 0, \\ u^2 + v^2 = 1. \end{cases}$

35. $\begin{cases} 9\,u^2 - y^2 = 16, \\ y^2 - 14 - 3\,u = 0. \end{cases}$

36. $\begin{cases} x^2 + xy + y^2 = 61, \\ xy = 29 - x - y. \end{cases}$

37. $\begin{cases} a^2 - b^2 = 6\,a + 4\,b - 10, \\ a - b = 2. \end{cases}$

38. $\begin{cases} x^2 + 4\,xy + y^2 - x - y = -22, \\ x^2 + y^2 = 25. \end{cases}$

39. $\begin{cases} x - 2\,y + 2\,z = 3, \\ y - 2\,x + 2\,z = 6, \\ x^2 + y^2 + z = 14. \end{cases}$

40. $\begin{cases} x^2 - y^2 - z^2 = 4, \\ x^2 - 2\,y^2 = 1, \\ x^2 + y^2 - 5\,z^2 = 8. \end{cases}$

41. $\begin{cases} 4\,x^2 - 5\,y^2 - 6\,z^2 = 22, \\ x^2 - y^2 + z^2 = -6, \\ 9\,x^2 + z^2 = 22. \end{cases}$

42. $\begin{cases} x^2 - 3\,xy + 5\,z = -1, \\ x - y + 2\,z = 1, \\ 2\,x + y + z = 8. \end{cases}$

Solve for x and y:

43. $\begin{cases} x + y = 2\,a, \\ x^2 + y^2 = 2\,a^2 + 2. \end{cases}$

44. $\begin{cases} x - 3\sqrt{xy} = 28, \\ x^{\frac{1}{2}}y^{\frac{1}{2}} + 4\,y = 8. \end{cases}$

45. $\begin{cases} 3\,x^2 - 5\,xy = (2\,a - b)(a + 7\,b), \\ x + y = 3(a - b). \end{cases}$

46. $\begin{cases} x^3 - 3\,x^{\frac{3}{2}}y^{\frac{3}{2}} + 2\,y^3 = 3, \\ 2\,x^3 + y^3 = 6. \end{cases}$

47. $\begin{cases} 3\,x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = -14, \\ x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} = 30. \end{cases}$

REVIEW EXERCISE 66

Problems on Chapters IX, X, XI and XII

Solve (a) by factoring; (b) graphically; (c) by the formula:

1. $3\,x^2 + x - 4 = 0$. **2.** $3 = 13\,x + 10\,x^2$. **3.** $4\,x^2 - 9 = 0$.

Graph each function:

4. $3\,x - 4\,x^2 - 2$. **5.** $2\,x^2 - 5\,x - 6$. **6.** $3\,x^2$.

Without solving, determine (a) the nature of the roots; (b) their sum and their product:

7. $5x^2 - 3x = 2$. **8.** $3x^2 + 9 = 4x$. **9.** $25x^2 = 10x - 1$.

10. Form a quadratic equation whose roots are the given numbers: (a) $\frac{3}{4}$ and $\frac{7}{5}$; (b) $\frac{1}{2}(3 - \sqrt{2})$ and $\frac{1}{2}(3 + \sqrt{2})$; (c) $(4 - 3i)$ and $(4 + 3i)$.

Determine the value of the constant h, under the given condition:

11. The roots are equal: $9x^2 - 6hx + 4h = 0$.

12. One root is zero: $5x^2 - 3hx + h^2 = 9$.

13. The sum of the roots is 2: $hx^2 - 3x + 2hx - 3 = 0$.

14. The product of the roots is 4: $hx^2 - 5x - 3h + 2 = 0$.

15. One root is the negative of the other:

$$3x^2 - 5x + 45h^2x = 6.$$

16. Solve for x by the method of completing the square:
(I) $4x^2 - 4x - 5 = 0$. (II) $3cx^2 - 5ax = b$.

17. If $f(x) = \left(3x - \frac{5}{x}\right)$, find $f(-3)$; $f(2)$; $f\left(\frac{a}{b}\right)$.

18. Solve and check: $2x^2 - 12x + 25 = 0$.

Solve graphically:

19. $\begin{cases} 4x^2 - 9y^2 = 36, \\ x^2 + 16y^2 = 16. \end{cases}$

20. $\begin{cases} x^2 - 4y^2 = 0, \\ x^2 + y^2 = 9. \end{cases}$

Solve:

21. $\sqrt{3+2x} + \sqrt{x+1} = 1$.

22. $3x^4 + 11x^2 - 20 = 0$.

23. $3x^{-4} + 23x^{-2} - 36 = 0$.

24. $\sqrt{x+7} - 2\sqrt{x} = -2$.

25. $\frac{3x+5}{3x+2} + \frac{2x+1}{2x} = 1$.

26. $\begin{cases} 3xy - 4x^2 = 2, \\ 6y^2 - 15xy + 10x^2 = 4. \end{cases}$

27. $\begin{cases} \frac{3}{x^2} + \frac{2}{y^2} = 66, \\ \frac{9}{x^2} + \frac{5}{y^2} = 189. \end{cases}$

28. $\begin{cases} x^2 + y^2 - x - 2y = 0, \\ x + 2y = 5. \end{cases}$

29. If a certain number is increased by three times its reciprocal, the sum is $12\frac{1}{4}$. Find the number.

30. A fast train runs 10 miles per hour faster than a slow train and takes two hours less to run 240 miles. Find the rates of the two trains.

31. By means of a graph, find the value of x for which the corresponding value of $(6x^2 - 8x - 5)$ is the least.

32. If $g(y) = 1 \div (1 - y)$, find $g(3)$; $g(a^2)$; $g\left(\frac{1}{b}\right)$.

33. Graph the function $f(x) = x^2 - 6x - 2$. From this graph, read the solutions (*a*) of $x^2 - 6x - 2 = 10$; (*b*) of $x^2 - 6x - 2 = 7$.

CHAPTER XIII

RATIO AND PROPORTION

107. Ratio. — The **ratio** of one number a to a second number b is the quotient $a \div b$. The *ratio of a to b* is also written $a : b$. Thus,

$$a : b = \frac{a}{b}. \tag{1}$$

NOTE. — A ratio is merely a fraction, and any fraction can be thought of as a ratio.

108. The ratio of two concrete quantities has meaning *only if they are of the same kind.* Their ratio is the quotient of their measures in terms of the same unit of measure.

Thus, the ratio of 3 tons to 500 pounds is 6000 : 500, because 3 tons contain 6000 pounds.

109. Proportion. — A **proportion** is a statement that two ratios are equal. That is, a proportion is merely a statement that two fractions are equal. The proportion

$$a : b = c : d \textit{ means that } \frac{a}{b} = \frac{c}{d}. \tag{1}$$

The proportion $a : b = c : d$ is read *a is to b as c is to d.*

NOTE. — The four numbers a, b, c, and d are said *to be in proportion* or, *to form a proportion.* The 1st and 4th numbers, a and d, are called the **extremes** and the 2d and 3d, b and c, are called the **means** of the proportion.

EXERCISE 67

Express each ratio as a fraction and simplify :

1. $3x : 2$. **2.** $\frac{5}{8} : \frac{3}{16}$. **3.** $(3a - \frac{1}{2}) : (36a^2 - 1)$. **4.** $2\frac{1}{4} : 8$.

Find the ratio of the given quantities:

5. 350 pounds to 5 tons. **6.** 51 pints to 82 quarts.

Change each equation to fractional form and solve:

7. $x : (25 - x) = 4$. **9.** $x : 3 = 5 : 27$.

8. $3x : (2x^2 - 5) = 2$. **10.** $5\frac{1}{3} : 2x = (3x - 1) : 5x$.

11. $(.3 - 2x) : 4 = (4 - .5x) : 3$.

12. $(4 - x) : (3 - x) = 2 : 5x$.

13. A line 15 inches long is divided into two parts whose lengths have the ratio 2 : 3. Find the lengths.

HINT. — Let x be the shorter length; $(15 - x)$ is the other. Then, $x : (15 - x) = 2 : 3$.

14. Divide 36 into two parts whose ratio is 4 : 5.

15. Divide 75 into two parts such that the ratio of one part, diminished by 4, to the other part, increased by 3, shall be 3 : 2.

16. The ratio of a first number to a second is 3 : 4, and the second number minus the first equals 28. Find the numbers.

NOTE. — If two triangles (or polygons of any number of sides) are similar, then (*a*) *the ratio of any two sides of one triangle equals the ratio of the corresponding sides of the other triangle*, and (*b*) *the area of one triangle is to the area of the other as the square of any side of the first triangle is to the square of the corresponding side of the other triangle.*

17. The sides of one triangle are 4, 7, and 11 inches. In a similar triangle, the longest side is 20 inches long. Find the other sides.

18. The sides of a certain polygon are 9, 6, 4, and 3 feet long. If the shortest side is increased by 1 foot, by how much must the other sides be lengthened to obtain a new polygon similar to the given one?

19. The area of a triangle, whose base is 15 inches long, is 190 square inches. Find the area of a similar triangle, whose base is 25 inches long.

20. The area of a certain quadrilateral is 36 square feet, and its shortest side is 3 feet long. Find the area of a similar quadrilateral whose shortest side is 5 feet long.

21. A man, 6 feet tall, stands 30 feet from a street light, and casts a shadow 10 feet long. How high above the ground is the light?

22. A man, $5\frac{1}{2}$ feet tall, stands beside a tree in the sunlight. The length of the shadow of the man is 8 feet and of the tree is 25 feet. How tall is the tree?

If $a : b = c : x$, then x is called the **fourth proportional** *to a, b, and c. Find the fourth proportional to each set of numbers:*

23. 2, -5, and 14.

24. 5, 4, and 2.

25. 2, b, and ab.

26. 3, z, and 5.

If $a : b = b : x$, then x is called the **third proportional** *to a and b. Find the third proportional to each pair of numbers:*

27. 18 and 50.

28. $2\frac{1}{3}$ and $\frac{3}{5}$.

29. $2a$ and a.

30. $12\,m^2n$ and $3\,mn^2$.

NOTE. — If $a : x = x : b$, then x is called a **mean proportional** between a and b. If $a : x = x : b$, then $x^2 = ab$; $x = \pm\sqrt{ab}$. Hence, if neither a nor b is zero, there are *two* mean proportionals between a and b.

Find the mean proportionals between

31. 125 and 5.

32. -3 and $-\frac{1}{27}$.

33. 216 and $\frac{1}{6}$.

34. 2 and -2.

110. Properties of proportions; supplementary. — The following properties are simple consequences of equation 1, Section 109, where we defined a proportion. The details of the proofs of the properties are left to the student.

I. *If* $\frac{a}{b} = \frac{c}{d}$, *then* $ad = bc$. Or, *in any proportion, the product of the means equals the product of the extremes.*

II. *If* $\frac{a}{b} = \frac{c}{d}$, *then* $\frac{a}{c} = \frac{b}{d}$. Or, *in any proportion, the means may be interchanged without destroying the proportion.*

NOTE. — In II, $a : c = b : d$ is said to be obtained by **alternation.**

III. *If* $\frac{a}{b} = \frac{c}{d}$, *then* $\frac{b}{a} = \frac{d}{c}$. Or, *in any proportion, the 2d term is to the 1st as the 4th term is to the 3d.*

NOTE. — In III, $b : a = d : c$ is said to be obtained by **inversion.**

IV. *If* $\frac{a}{b} = \frac{c}{d}$, *then* $\frac{a + b}{b} = \frac{c + d}{d}$.

HINT for proof. — Add 1 to both sides of the given equation and reduce each member to a single fraction.

NOTE. — In IV, $(a + b) : b = (c + d) : d$ is said to be obtained by **composition.**

V. *If* $\frac{a}{b} = \frac{c}{d}$, *then* $\frac{a - b}{b} = \frac{c - d}{d}$.

HINT for proof. — Subtract 1 from each side of the given equality.

NOTE. — In V, the proportion $(a - b) : b = (c - d) : d$ is said to be obtained by **division.**

VI. *If* $\frac{a}{b} = \frac{c}{d}$, *then* $\frac{a + b}{a - b} = \frac{c + d}{c - d}$.

HINT for proof. — Divide each side of $\frac{a + b}{b} = \frac{c + d}{d}$, from IV, by the corresponding side of $\frac{a - b}{b} = \frac{c + d}{d}$, from V.

NOTE. — The new proportion in VI is said to be obtained from $a : b = c : d$ by **composition and division.**

EXERCISE 68

From each proportion, obtain five new proportions by applying properties II *to* VI:

1. $2 : m = 5 : x$.

2. $a : b = x : y$.

3. $3 : 7 = c : y$.

4. $c^2 : d = x : b^2$.

From the proportion $c : x = d : y$, *obtain a new proportion by applying the specified properties :*

5. *Inversion*, and, to the result, apply *composition*.

6. *Alternation*, and, to the result, apply *division*.

If $a : b = c : d$, *prove each of the following by use of* II *to* VI :

7. $(a + b) : (c + d) = b : d$. 8. $(a - b) : (c - d) = b : d$.

HINT for Problem 7. — First use composition.

11. $\dfrac{2a + 3b}{2c + 3d} = \dfrac{2a - 3b}{2c - 3d}$. 12. $\dfrac{a + b}{a} = \dfrac{c + d}{c}$.

HINT for Problem 11. — Since $\dfrac{a}{b} = \dfrac{c}{d}$, then $\dfrac{2a}{3b} = \dfrac{2c}{3d}$.

13. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, prove that $\dfrac{a + c + e}{b + d + f} = \dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$.

HINT. — Let $k = \dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$. Then, $a = kb$; $c = kd$; $e = kf$.

CHAPTER XIV

THE LANGUAGE OF VARIATION

111. Direct Variation. — Each of the statements

y varies directly as x, or, **y varies as x**, (1)

y is directly proportional to x, or, **y is proportional to x**, (2)

means that there exists a constant k such that, for every value of x there is a corresponding value of y, given by

$$y = kx. \tag{3}$$

We call k the **constant of variation**, or **of proportionality.**

Thus, the circumference C of a circle varies directly as the radius r because $C = 2\pi r$. Here, the constant of proportionality is 2π.

NOTE. — The symbol $\propto$ is sometimes used to abbreviate any one of the statements in (1) and (2). Thus, $y \propto x^2$ means that y *varies as* x^2.

NOTE. — Colloquially, *to vary* means *to change*, but recognize that we have introduced a distinctly different technical meaning for *varies* as used in statement 1 of this section. Thus, if y is any function of x, then y *changes as* x *changes*, but, we do not say that y *varies as* x unless y is the particularly simple function $y = kx$.

112. Inverse variation. — Each of the statements

y is inversely proportional to x, or, **y varies inversely as x**, (1)

means that there is a constant k such that, for every value of x, the corresponding value of y is given by

$$y = \frac{k}{x}.$$

Thus, to say that the time t, necessary for a train to go a certain distance d miles, varies inversely as the speed s, means that $t = \frac{k}{s}$.

NOTE 1. — If $y = \frac{k}{x}$, then $y = k\left(\frac{1}{x}\right)$, or y *is proportional to* $\frac{1}{x}$. Thus, $y \propto \frac{1}{x}$ may be used to abbreviate either statement in (1).

113. Joint Variation. — To say that y **varies jointly** as x and z means that y is directly proportional to the product xz, or that

$$y = kxz. \tag{1}$$

Thus, the simple interest earned by a given principal P varies jointly as the time t and the rate r, because $I = Prt$. In this instance, the constant of variation (k, in equation 1) is P.

114. The various types of variation may be combined. Thus, to say that y *varies directly as* xz *and inversely as* w^3 means that $y = (kxz) \div w^3$.

Example 1. — If y is directly proportional to x and varies inversely as w^2, and if $y = 2$ when $x = 1$ and $w = 3$, find y when $x = 2$ and $w = 1$.

Solution. — 1. From the given conditions $y = \frac{kx}{w^2}$. (1)

2. To find the constant of variation, substitute $x = 1$, $y = 2$, and $w = 3$ in (1): $2 = \frac{k}{9}$; $k = 18$. Hence, $y = \frac{18x}{w^2}$.

3. Substitute $x = 2$ and $w = 1$ in the result of Step 2: $y = 36$.

EXERCISE 69

In each of Problems 1 *to* 10, *express the relationship between the variables by an equation, and insert the value of the constant of variation, when it can be found:*

1. w varies directly as y and inversely as z.
2. A is proportional to x and varies inversely as z^3.
3. y varies as x, and $y = 3$ when $x = 1$.
4. B varies as C and $B = 16$ when $C = \frac{3}{4}$.

5. y is inversely proportional to x^2, and $y = -2$ when $x = 3$.

6. y is inversely proportional to z, and directly proportional to x, and $y = 5$ when $x = 15$ and $z = 30$.

7. $(y + 3)$ varies as $(2x + 3)$, and $x = 1$ when $y = 3$.

8. $(3z + 1)$ varies as $(5x - 1)$, and $z = 2$ when $x = \frac{2}{5}$.

9. The weight of a body above the surface of the earth varies inversely as the square of the distance of the body from the center of the earth.

10. The volume of a circular cone varies jointly as the altitude and the square of the radius of the base.

For each of the following formulas, give an equivalent statement in the language of variation:

11. $w = \dfrac{10\,r^3}{z}$. **12.** $A = \frac{1}{2}\,ab$. **13.** $y \propto \dfrac{w}{z^2}$. **14.** $z = \dfrac{4\sqrt{x}}{y}$.

Solution of Problem 11. — The value of w is directly proportional to r^3 and inversely proportional to z.

15. If y is proportional to x, and varies inversely as $\sqrt{w}$, and if $y = 7$ when $x = 2$ and $w = 4$, find the value of y when $x = 1$ and $w = 9$.

16. The number of posts required for a fence varies inversely as the distance between them. If it takes 80 posts when they are placed 12 feet apart, how many will be required when they are placed 15 feet apart?

17. The distance fallen by a body from a position of rest varies as the square of the number of seconds occupied in falling. If a body falls 256 feet in 4 seconds, how far will it fall in 6 seconds?

18. The surface of a cube varies as the square of its edge. If the surface of a cube, whose edge is $2\frac{1}{3}$ inches, is $32\frac{2}{3}$ square inches, what will be the edge of a cube whose surface is $30\frac{3}{8}$ square inches?

19. The illumination from a source of light varies inversely as the square of the distance from the light. If a book, now 10 inches away from the light, is moved to 20 inches distance, what is the ratio of the new illumination to the old?

HINT. — The value of the constant of variation cannot be found from the data and is not needed. Compute the quotient of the illumination at 20 inches distance divided by the illumination at 10 inches distance.

20. Newton's *Law of Gravitation* states that the force with which each of two masses of m pounds and M pounds attracts the other varies directly as the product of the masses and inversely as the square of the distance between the masses. Find the ratio of the force of attraction when two masses are 4000 miles apart to the force when they are 2000 miles apart.

21. If y varies directly as x, and if $y = 16$ when $x = 2$, determine the equation relating x and y, and draw its graph.

22. If y varies inversely as x, and if $y = 15$ when $x = 5$, determine the equation relating x and y, and draw its graph.

23. A law in Astronomy known as *Kepler's Third Law of Planetary Motion* states that *the square of the mean distance of any planet from the sun varies directly as the cube of the time it takes the planet to make one complete revolution about the sun.* The mean distance of the earth is 93,000,000 miles. Find the time in years for a complete revolution (*a*) of Venus, given that its mean distance is 67,000,000 miles; (*b*) of Mars, given that its mean distance is 141,500,000 miles.

24. When an image of an astronomical object is formed in the focus of a telescope, the brightness of the image varies directly as the square of the diameter of the lens of the telescope and inversely as the square of the distance from the image to the lens. In one telescope the diameter of the lens is 40 inches and the image is 60 feet from the lens. In another telescope, the lens is 20 inches in diameter and the image is 40 feet from the lens. What is the ratio of the brightness of the image in the first telescope to that in the second?

SUPPLEMENTARY PROBLEMS

NOTE 1. — Either of the statements (1) *x is to y is to z as r is to s is to t*, or (2) *x, y, and z are proportional to r, s, and t, respectively*, means that there exists *one* constant k such that $x = kr$, $y = ks$, and $z = kt$. Either of statements (1) and (2) is abbreviated by writing $x : y : z = r : s : t$.

Example 1. — Divide 395 into three parts proportional to 5, 3, and 2.

Solution. — 1. Let x, y, and z be the parts. Then, $x : y : z = 5 : 3 : 2$, and $x + y + z = 395$.

2. By the preceding Note 1, $x = 5k$; $y = 3k$; $z = 2k$.

3. Since $x + y + z = 395$, $5k + 3k + 2k = 395$; $10k = 395$.

4. Hence, $k = 39.5$; therefore $x = 5k = 177.5$; $y = 118.5$; $z = 79$.

Find x, y, and z under the given conditions:

25. $x : y : z = 3 : -2 : 4$; $x + 2y + z = 20$.

26. $x : y : z = 5 : 3 : -3$; $x - y - z = 10$.

27. $x : y : z = 1 : 5 : 2$; $x^2 + y^2 + z^2 = 1$.

28. $x : y : z = -2 : 0 : 4$; $x^2 + y^2 + z^2 = 1$.

29. A farm, of 2750 acres, is divided into three parts proportional to 8, 2, and 1 respectively. Find the area of each part.

30. Separate 250 into three parts which are to each other as 3 : 2 : 5.

CHAPTER XV

PROGRESSIONS

115. Arithmetic progressions. — A **sequence** of things is a set of things arranged in a definite order. An **arithmetic progression** is a sequence of numbers called **terms**, each of which, after the first, is derived *from the preceding one by adding to it a fixed number* called the **common difference.**

Thus, in the arithmetic progression 9, 6, 3, 0, $-3, \cdots$, the common difference is -3. The 6th term would be -6.

NOTE. — We shall abbreviate the words *arithmetic progression* by the letters A.P. The common difference in an A.P. can be obtained by subtracting any term from the one following it.

116. The *n*th term in an A.P. — Let a be the 1st term and d the common difference of an A.P. Then, the 2d term is $(a + d)$; the 3d term is $(a + 2d)$; the 4th term is $(a + 3d)$. The coefficient of d in each of these terms is 1 *less than the number of the term*. Similarly, we find that the 10th term is $(a + 9d)$. The nth term is the $(n - 1)$st after the 1st term, and is obtained after d has been added $(n - 1)$ times, in succession. Hence, if l represents the nth term,

$$l = a + (n - 1)d. \tag{1}$$

Thus, if $a = 3$ and $d = -2$, the 17th term is $3 + 16(-2) = -29$.

117. The sum of the first *n* terms of an A.P. — Let S be the sum of the first n terms of the progression considered in Section 116. The last term is l; the next to the last is

$(l - d)$, etc. Writing the sum of the n terms, forward and backward, we obtain

$$S = a+(a+d)+(a+2\,d)+ \cdots +(l-2\,d)+(l-d)+ l, \quad (1)$$

$$S = l+(l-d)+(l-2\,d)+ \cdots +(a+2\,d)+(a+d)+a. \quad (2)$$

On adding corresponding sides of (1) and (2) we obtain

$$2\,S = (a+l)+(a+l)+(a+l)+ \cdots +(a+l)+(a+l)+(a+l),$$

or, $$2\,S = n(a + l).$$

Hence, $$\boldsymbol{S = \frac{n}{2}(a + l).} \quad (3)$$

NOTE. — Substitute $l = a +(n - 1)d$ in (3); then,

$$\boldsymbol{S = \frac{n}{2}[2\,a + (n - 1)d].} \quad (4)$$

Example 1. — Find the sum of the A.P. $8 + 5 + 2 + \cdots$ to twelve terms.

First Solution. — 1. Use $l = a +(n - 1)d$: $l = 8 + 11(-3) = -25$.

2. From formula 3, Section 117, $S = 6(8 - 25) = -102$.

Second Solution. — From formula 4, $S = 6[16 + 11(-3)] = -102$.

NOTE. — The quantities a, d, l, n, and S are called the **elements** of the general arithmetic progression. If any three of the elements are given, the other two can be found by use of the following equations:

$$\boldsymbol{l = a +(n - 1)d;\ S = \frac{n}{2}(a + l).} \quad (5)$$

EXERCISE 70

By use of $l = a +(n - 1)d$, find the specified terms of the A.P. *with the given terms:*

1. Given terms: 2, 6, 10; find the 9th term; the 61st term.

2. Given terms: 2, 2.05, 2.10; find the 18th term.

3. Given terms: $-3, -8, -13$; find the 36th term.

4. Given terms: $4, -1, -6$; find the 17th term.

Find the last term and the sum of each A.P.:

5. 4, 9, 14, ··· to 14 terms.

6. 8, 6, 4, ··· to 12 terms.

7. 6, 1, − 4, ··· to 18 terms.

8. 2, 4, 6, ··· to 31 terms.

9. 1.08, 1.16, ··· to 22 terms.

10. $\frac{5}{2}$, 2, $\frac{3}{2}$, ··· to 41 terms.

In each problem, certain of the elements a, d, l, n, and S are given; find the missing ones by use of formulas 5, *Section* 117:

11. $a = 3, l = 403, n = 26$.

12. $a = 5, l = 395, d = 5$.

13. $n = 15, a = 7, l = 73$.

14. $l = 57, d = 4, n = 17$.

15. $a = \frac{2}{3}, n = 16, S = 2\frac{2}{3}$.

16. $n = 20, a = 8, l = -9.1$.

Definition. — *In an A.P., the terms between any two other terms are called* **arithmetic means** *between those two terms.*

Example. — Insert 5 arithmetic means between 13 and − 11.

Explanation. — We wish to find five numbers which, when placed in sequence between 13 and − 11, will complete an A.P. of seven terms with 13 as the first and − 11 as the last term. To find the five means, first find d for the A.P. and then form the terms.

Solution. — 1. We are given $a = 13$, $n = 7$, and $l = -11$.

2. Use $l = a + (n - 1)d$: $-11 = 13 + 6d$; $6d = -24$; $d = -4$.

3. The A.P. is 13, 9, 5, 1, − 3, − 7, − 11; (9, 5, 1, − 3, − 7) are the means.

17. Insert four arithmetic means between 2 and − 10.

18. Insert three arithmetic means between 1 and 7.

19. Insert seven arithmetic means between $\frac{3}{2}$ and 6.

20. Insert four arithmetic means between − 15 and 14.

21. Insert ten arithmetic means between h and k.

Definition. — *When a* ***single*** *arithmetic mean is inserted between two numbers, this is called* ***the*** *arithmetic mean of the numbers.*

22. Find the arithmetic mean (*a*) of 5 and 13; (*b*) of − 3 and 12; (*c*) of − 56 and − 32; (*d*) of x and y.

23. In the A.P. 1.05, 1.10, 1.15, ···, which term is 2.25?

24. In the A.P. 7, 4, 1, ···, which term is − 80?

25. What is the common difference of an A.P. in which the first term is 6 and the 27th term is 188?

26. The 5th term of an A.P. is 430, and the 45th term is 110; find the 2d term.

HINT. — Think of the 5th term as the 1st term of an A.P.; then find d.

NOTE. — In the solution of a word problem involving a progression, (1) write down the first few terms of the progression, describing them in the language of the problem; (2) decide which of the elements of the progression are known and which you wish to find.

27. Find the sum of all odd integers from 1 to 207, inclusive.

28. Find the sum of all even integers from 8 to 570, inclusive.

29. Find the sum of the first 57 positive integral multiples of 3.

30. There are 20 layers of posts in a pile. The first layer contains 40 posts, and each succeeding layer contains one post less than the preceding layer. How many posts are in the pile?

31. In a certain school system, a teacher is paid \$1250 the first year, and is given an increase of \$50 per year each year thereafter. (*a*) What is the teacher's income during the 22d year? (*b*) What is his total income during the first 23 years?

32. A man borrows \$4000. He agrees to pay \$200 of the principal at the end of each year, as long as necessary. Also, he agrees to pay, at the end of each year, 5% interest on the principal outstanding during that year. What is the total sum which he pays to his creditor in discharging his debt.

33. In each year after the first, a man saves \$50 more than in the preceding year and by the end of 20 years he has saved \$25,000. What did he save in the first year?

118. A geometric progression is a sequence of numbers called **terms**, each of which, after the first, is obtained by *multiplying the preceding term by a fixed number* called the **common ratio.**

Thus, in the geometric progression $16, -8, +4, -2, \cdots$, the common ratio is $-\frac{1}{2}$; the 5th term would be $(-\frac{1}{2})(-2) = +1$.

NOTE. — In a geometric progression, the common ratio equals *the ratio of any term,* after the first, *to the one preceding it.*

When desirable, we shall abbreviate the words *geometric progression* by the letters **G.P.**

119. The *n*th term of a G.P. — If a represents the 1st term, and r the common ratio, then the 2d term is ar; 3d term is ar^2; 4th term is ar^3. In each of these terms the exponent of r is **1 less than the number of the term**. Similarly, we find that the 8th term is ar^7. The nth term is the $(n-1)$st after the 1st term and hence is obtained by multiplying a by $(n-1)$ factors r, or by r^{n-1}. Therefore, if l represents the nth term,

$$l = ar^{n-1}. \tag{1}$$

Thus, if $a = 3$ and $r = 2$, the 7th term is $3(2^6) = 192$.

120. The sum of a G.P. — Let s represent the sum of the first n terms of the G.P. considered in Section 119. The terms are

$$a,\ ar,\ ar^2,\ \cdots,\ ar^{n-2},\ ar^{n-1}, \tag{1}$$

where ar^{n-2} is the $(n-1)$st term. Hence,

$$s = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}; \tag{2}$$

$$sr = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n, \tag{3}$$

where (3) is obtained by multiplying both sides of (2) by r. On subtracting each side of (3) from the corresponding side of (2), we obtain

$$s - sr = a - ar^n, \tag{4}$$

because each term, except ar^n, on the right in (3) cancels a corresponding term in (2). From (4), $s(1-r) = (a - ar^n)$, or

$$s = \frac{a - ar^n}{1 - r}. \tag{5}$$

Since $l = ar^{n-1}$, then $rl = ar^n$. On replacing ar^n by rl in (5), we obtain

$$s = \frac{a - rl}{1 - r}. \tag{6}$$

NOTE. — The numbers (a, r, n, l, s) are called the elements of the general geometric progression, and are connected by the formulas

$$l = ar^{n-1};\ s = \frac{a - rl}{1 - r};\ s = \frac{a - ar^n}{1 - r}. \tag{7}$$

Example. — Find the sum of the G.P.

$$2,\ 6,\ 18,\ \cdots \text{ to six terms.}$$

Solution. — Use formula 5 with $n = 6;\ a = 2;\ r = 3$:

$$s = \frac{2 - 2 \cdot 3^6}{1 - 3} = \frac{2 - 1458}{-2} = 728.$$

EXERCISE 71

Use $l = ar^{n-1}$ to find the specified terms of the G.P.

1. Given terms: 1, 2, 4: find the 8th and the 12th terms.

2. Given terms: 4, − 12, 36: find the 11th term.

3. Given terms: $-\frac{3}{2}$, 3, − 6: find the 7th term.

By use of formulas 7, find the last term and the sum of each G.P:

4. 6, 12, 24, ⋯ *to 7 terms.*

5. 20, 10, 5, ⋯ *to 8 terms.*

6. 4, − 12, 36, ⋯ *to 6 terms.*

7. 5, .5, .05, ⋯ *to 7 terms.*

8. − 4, 2, − 1, ⋯ *to 5 terms.*

9. $4,\ 4b^2,\ 4b^4, \cdots$ *to 15 terms.*

10. $a,\ ab^4,\ ab^8, \cdots$ *to 30 terms.*

11. $\frac{1}{36},\ -\frac{1}{12},\ \frac{1}{4}, \cdots$ *to 6 terms.*

Find an expression for the sum of each G.P. *by use of* (6):

12. $1 + (1.05) + (1.05)^2 + \cdots + (1.05)^{20}$.

13. $(1.04)^{\frac{1}{2}} + (1.04) + (1.04)^{\frac{3}{2}} + \cdots + (1.04)^{26\frac{1}{2}}$.

14. $(1.06)^{-25} + (1.06)^{-24} + (1.06)^{-23} + \cdots + (1.06)^{-2}$.

15. $5(1.03)^{-4} + 5(1.03)^{-6} + 5(1.03)^{-8} + \cdots + 5(1.03)^{-40}$.

16. $(1 + r) + (1 + r)^2 + (1 + r)^3 + \cdots + (1 + r)^n$.

In each problem, certain of the elements (a, l, r, n, s) *are given; find the missing elements by use of formulas* 7, *page* 163:

17. $s = 750;\ r = 2;\ l = 400.$

18. $s = 3630;\ a = 30;\ l = 2430.$

19. $s = 255;\ l = 1;\ r = \frac{1}{2}.$

20. $s = 1,111,111;\ r = 10;\ a = 1.$

21. If the common ratio of a G.P. is 5 and if the 5th term is 1250, find the 1st term.

22. Which term of the G.P. 5, 10, 20, ···, is 640?

23. Find the common ratio of a G.P. whose 1st term is 4 and 5th term is $\frac{4}{81}$.

24. Find the common ratio of a G.P. whose 4th term is $\frac{2}{9}$ and 9th term is 54.

Definition. — In a G.P., *the terms between any two terms are called* **geometric means** *between those terms.*

Example. — Insert two geometric means between 6 and $\frac{16}{9}$.

Solution. — 1. After the means are inserted, they will complete a G.P. of four terms with $a = 6$ and $l = \frac{16}{9}$. We must find the common ratio of this G.P. and its two middle terms.

2. Use $l = ar^{n-1}$: $\frac{16}{9} = 6r^3;\ r^3 = \frac{8}{27};\ r = \frac{2}{3}$.

3. The G.P. is 6, 4, $\frac{8}{3}$, $\frac{16}{9}$; the geometric means are 4 and $\frac{8}{3}$.

Note. — In all cases, when asking for geometric means we shall desire *real valued* means. Hence, in solving $r^3 = \frac{8}{27}$ in the preceding example, we used only the real solution, $r = \frac{2}{3}$.

25. Insert five geometric means between 1 and 64.

26. Insert two different sets of three geometric means between 1 and 81.

27. Between $\frac{1}{16}$ and 4, insert (*a*) two geometric means; (*b*) five geometric means.

28. Insert four geometric means (*a*) between $\frac{1}{3}$ and 81; (*b*) between 81 and $\frac{1}{3}$.

Definition. — *If x and y are of the same sign, and if a single geometric mean G of the same sign is inserted between x and y, then G is called* **the** *geometric mean of x and y.*

29. Find the geometric mean (a) of 9 and 81; (b) of x and y.

30. Insert three geometric means between a and b.

31. If the 7th term of a G.P. is 4, and the 11th term is $\frac{4}{81}$, find the intermediate terms.

MISCELLANEOUS PROBLEMS IN ARITHMETIC AND GEOMETRIC PROGRESSIONS

1. If a man deposits \$.10 in a savings bank on January 1, and if, on the 1st of each succeeding month, he deposits twice as much as in the preceding month, how much will he have deposited by the end of 1 year?

2. At a bazaar, tickets are marked with the consecutive even integers 2, 4, 6 $\cdots$ and are drawn at random by those entering the bazaar. If each person pays a number of cents equal to the number on his ticket, how much money is received if 1000 tickets are marked and sold?

3. A rubber ball is dropped from a height of 50 feet. If, on each rebound, the ball rises to $\frac{2}{3}$ of the height from which it last fell, what distance has the ball traversed up to the instant it hits the ground for the 10th time?

4. A merchant's investment yields him, each year after the first, twice as much as in the preceding year. If his investment paid him \$7500 in the first four years, how much did he realize in the 1st and in the 4th years?

5. A 100-gallon cask is filled with pure alcohol. Twenty gallons are drawn off, and replaced by water; then, 20 gallons of the mixture are drawn off, and replaced by water, etc., until 10 drawings and replacements of 20 gallons each have been made. How much alcohol is present in the final mixture?

6. There are 20 equidistant balls in a straight line. A person starts from a position in a line with and in front of the

balls, his distance from the first one being twice the distance between adjacent balls. He walks to the first ball, picks it up and returns with it to his original position. He repeats the process, separately for each ball, and finds that he walks 6300 feet in bringing the 20 balls to his original position. Find the distance between adjacent balls.

7. A stock market speculator starts with \$10,000 capital. At the end of 1 year his gains on transactions total \$10,000 and his losses \$1000. In each succeeding year, his gains increase by \$100, and his losses by \$1100, as compared with the previous year. (*a*) When will his capital be greatest? (*b*) When will his capital become zero?

8. At the beginning of each year, a man invests \$300 at simple interest at the rate 5%. At the end of 20 years, what is the accumulated value of his investments, if none of them have been disturbed since they were made?

9. If 1000 units of a commodity are consumed in the 1st year and if the annual rate of increase of consumption is 6%, find the total consumption during the first 20 years, given that $(1.06)^{20} = 3.2071$.

HINT. — If P is the consumption in any year, the consumption in the next year is $P + P(.06) = P(1.06)$.

10. A man, climbing a mountain, ascends 1000 feet in the 1st hour, and 100 feet less in each succeeding hour than in the preceding hour. How long will he take to climb 5200 feet?

11. If the numbers 1, 1, 3, and 9 are added, respectively, to the first four terms of a certain A.P., a G.P. results. Find the first term and the common difference of the A.P.

HINT. — Use the fact that, in a G.P., the ratio of any term to the preceding one is a constant.

12. Find the sum of the first 30 positive integral powers of (1.07), given that $(1.07)^{15} = 2.7590$.

13. The 6th term of a G.P. is $\frac{4}{27}$ and the 11th term is $-\frac{128}{6561}$. Find the 3d term.

14. \$2000 is divided between eight persons so that each, after the first one, receives \$10 more than the preceding one. How much does each person receive?

15. The number of inhabitants in a certain city increased from 128,000 to 250,000 in six years. Assuming that the rate of increase was uniform, find the annual rate per cent of increase.

16. Find the sum (*a*) of the first 20 positive integral multiples of 5; (*b*) of the first n positive integral multiples of 5.

17. Find five numbers, forming an A.P., such that their sum shall be 25 and the sum of their squares 135.

18. If x, a, y form an A.P., prove that $a = \frac{1}{2}(x + y)$.

19. If x, g, and y form a G.P., prove that $g^2 = xy$.

20. In a certain positive integral number of three digits, the digits form an A.P. If the number is divided by the sum of its digits, the quotient is 48. When the three digits are written in reverse order, they form a number 396 less than the original number. Find the original number.

SUPPLEMENTARY TOPICS

121. Harmonic progressions. — A sequence of numbers is said to form a **harmonic progression** if the reciprocals of the numbers form an arithmetic progression.

Thus, 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ is a harmonic progression because the reciprocals, 1, 2, 3, 4, 5, form an arithmetic progression.

NOTE. — Suppose that a set of strings of the same diameter and substance are stretched to uniform tension. If the lengths of the strings form a harmonic progression, a harmonious sound results if two or more strings are caused to vibrate at one time. This fact accounts for the name *harmonic progression.*

In a harmonic progression, the terms lying between any two other terms are called **harmonic means** between those two terms. To find k *harmonic means* between two numbers a and b, first find k *arithmetic means* between $\frac{1}{a}$ and $\frac{1}{b}$; the

harmonic means are the reciprocals of these arithmetic means. If one harmonic mean is inserted between x and y, it is called **the** harmonic mean of x and y.

Example 1. Insert five harmonic means between 4 and 16.

Solution. — 1. First, we seek five arithmetic means between $\frac{1}{4}$ and $\frac{1}{16}$. Let d be the common difference of the A.P. of 7 terms, with $a = \frac{1}{4}$ and $l = \frac{1}{16}$.

2. Use the formula $l = a + (n-1)d$: $\frac{1}{16} = \frac{1}{4} + 6d$; $d = -\frac{1}{32}$.

3. The arithmetic progression is $\frac{1}{4}, \frac{7}{32}, \frac{6}{32}, \frac{5}{32}, \frac{4}{32}, \frac{3}{32}, \frac{1}{16}$.

4. The arithmetic means between $\frac{1}{4}$ and $\frac{1}{16}$ are: $\frac{7}{32}, \frac{6}{32}, \frac{5}{32}, \frac{4}{32}, \frac{3}{32}$.

5. The harmonic means between 4 and 16 are: $\frac{32}{7}, \frac{32}{6}, \frac{32}{5}, 8, \frac{32}{3}$.

EXERCISE 72

1. Insert 3 harmonic means between $\frac{1}{2}$ and $\frac{1}{10}$.

2. Insert 5 harmonic means between 6 and 24.

3. Insert 4 harmonic means between $\frac{5}{12}$ and $\frac{5}{2}$.

4. Insert 7 harmonic means between 16 and 5.

5. Find the harmonic mean (*a*) of 7 and 25; (*b*) of x and y.

6. If A, G, and H represent, respectively, the arithmetic, geometric, and harmonic means of x and y, prove that G is the geometric mean of A and H; that is, show that $G^2 = AH$.

7. If a, b, c, and d form an harmonic progression, prove that $(ab) \div (cd) = (a - b) \div (c - d)$.

122. Geometric progressions with infinitely many terms; supplementary. — Let S_n be the sum of the first n terms of a G.P. Then, by formula 5, Section 120,

$$a + ar + ar^2 + \cdots + ar^{n-1} = S_n = \frac{a}{1-r} - \frac{ar^n}{1-r}. \quad (1)$$

Illustration 1. Consider the G.P.

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \cdots \textit{ to infinitely many terms.} \quad (2)$$

In (2), $a = 1$, $r = \frac{1}{2}$, and the nth term is $\frac{1}{2^{n-1}}$; $(1 - r) = \frac{1}{2}$ and $ar^n = \frac{1}{2^n}$. Hence, by (1),

$$S_n = 2 - \frac{1}{2^{n-1}}. \tag{3}$$

As n grows larger, without limit, the term $\frac{1}{2^{n-1}}$ in (3) grows smaller, and is as near to zero as we please, if n is sufficiently large. Thus, if $n = 65$, $\frac{1}{2^{n-1}} = \frac{1}{2^{64}} = \frac{1}{18,446,744,073,709,551,616}$, which is practically zero. Hence, in (3), for all values of n which are sufficiently large, S_n will be as near to $(2 - 0)$ as we please. To summarize this statement we say that ***as n becomes infinite***, S_n approaches the **limit** 2, and we call 2 the **sum** of the progression $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ *to infinitely many terms.*

The symbols "$n \to \infty$" are used to abbreviate the phrase "*n becomes infinite.*" Then, in Illustration 1 our conclusion can be briefly written $\underset{n \to \infty}{\textit{limit}}\, S_n = 2$.

In general, consider any infinite G.P.

$$a, ar, ar^2, \cdots \textit{to infinitely many terms}, \tag{4}$$

in which r *is numerically less than* 1, that is, in which r *is between* -1 *and* $+1$. As n grows large without bound, the numerator ar^n in (1) grows smaller, and is as near to zero as we please, for all values of n sufficiently large. Hence, from (1) we see that, as n grows large without limit, the value of S_n approaches

$$\frac{a}{1 - r} - \frac{0}{1 - r}, \text{ or } \frac{a}{1 - r}.$$

That is, the limit of S_n, as n becomes infinite, is $\frac{a}{1 - r}$:

$$\underset{n \to \infty}{\textit{limit}}\, S_n = \frac{a}{1 - r}. \tag{5}$$

This limiting value of the sum of n terms, as n becomes infinite, is called the **sum** of the G.P. with infinitely many terms. If S represents this sum, then

$$S = \frac{a}{1 - r}. \tag{6}$$

NOTE. — S is not a *sum* in the ordinary sense of the word. It is *the limit of the sum of n terms as n grows large without limit.*

Illustration 2. By formula (6), the sum of the progression

$$5, \tfrac{5}{2}, \tfrac{5}{4}, \cdots \text{ to infinitely many terms,} \tag{7}$$

is $S = \dfrac{5}{1 - \frac{1}{2}} = 10$. Practically, this means that *if we add a relatively large number of terms of* (7), *we will obtain approximately* 10, and that *by adding enough terms, we can obtain a result as close to* 10 *as we may desire.* Thus, the sum of the first 11 terms is $S_{11} = 9\frac{1023}{1024}$.

NOTE. — If the ratio r of a G.P. is numerically *greater than* or *equal to* 1, then S_n does *not* approach a limit as n becomes infinite. Thus, for the G.P. 1, 2, 4, 8, $\cdots$, S_n increases beyond all bounds as n becomes infinite.

Example 1. — Find a rational fraction equal to the unending repeating decimal .5818181 $\cdots$.

Solution. — 1. $.5818181 \cdots = 5 + .081 + .00081 + .0000081 + \cdots$.

2. We notice that the terms in $.081 + .00081 + \cdots$ form a G. P. with infinitely many terms, with $r = .01$. Hence, by (6), the sum of this progression is $\dfrac{.081}{1 - .01} = \dfrac{.081}{.99} = \dfrac{9}{110}$.

Hence, $.5818181 \cdots = .5 + \frac{9}{110} = \frac{5}{10} + \frac{9}{110} = \frac{64}{110} = \frac{32}{55}$.

NOTE. — By the method of Example 1, we can show that any repeating decimal is equal to a rational fraction.

EXERCISE 73

Find the sum of each of the following progressions with infinitely many terms:

1. $6, 2, \frac{2}{3}, \cdots$
2. $16, 4, 1, \cdots$
3. $1, -\frac{1}{5}, \frac{1}{25} \cdots$
4. $1, \frac{1}{4}, \frac{1}{16}, \cdots$
5. $5, \frac{5}{10}, \frac{5}{100}, \cdots$
6. $1, -.1, .01, -.001, \cdots$

7. A rubber ball is dropped from a height of 100 feet. On each rebound the ball rises to $\frac{2}{3}$ of the height from which it last fell. Find the distance traversed by the ball in coming to rest.

8. A pendulum is being brought to rest by air resistance; the path of each swing (after the first) of the pendulum bob is .99 as long as that of the previous swing. If the path of the first swing is 20 inches long, find the total distance traveled by the bob in coming to rest.

Find the rational fraction which equals the given unending repeating decimal, where repeating parts are written three times:

9. .666 ···
10. .444 ···
11. 2.010101 ···
12. .4090909 ···
13. .24333 ···
14. .5222 ···
15. .636363 ···
16. .08333 ···
17. .1666 ···
18. 1.4212121 ···
19. .589111 ···
20. 1.45222 ···

CHAPTER XVI

MATHEMATICAL INDUCTION *

123. Mathematical induction. — Many theorems in mathematics can be proved by a method called **mathematical induction**, which is illustrated in proving

Theorem I. — *If n is any positive integer, then* †

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}. \qquad (1)$$

Proof; Part I. — From (1) we obtain the following results:

When we place $n = 1$, $\quad 1 = \frac{1(1+1)}{2}$, or $1 = 1$.

When we place $n = 2$, $\quad 1 + 2 = \frac{2(2+1)}{2}$, or $3 = 3$.

When we place $n = 3$, $\quad 1 + 2 + 3 = \frac{3(3+1)}{2}$, or $6 = 6$.

Hence, equation 1 is true when n is 1, 2, or 3. The verification of these special cases creates a presumption that *all* cases of the theorem are true, but, *by itself, the verification of any number of special cases does not prove a general theorem.*

Part II; Auxiliary Theorem. — *If k is any value of n for which equation* 1 *is true, then it is true also when $n = k + 1$.*

By hypothesis, (1) is true when $n = k$, or

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}. \qquad (2)$$

* This chapter is a prerequisite only for certain supplementary sections in later chapters.

† When n is 5, $(1 + 2 + 3 + \cdots + n)$ means $(1 + 2 + 3 + 4 + 5)$.

By use of (2), we wish to prove that

$$1+2+3+\cdots+(k+1) \stackrel{?}{=} \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}, \quad (3)$$

which is the equation resulting from (1), when $n = (k + 1)$. In (3) we place " ? " over " = " because the equality is not yet proved. Recognize that

$$1 + 2 + 3 + \cdots + (k + 1) = 1 + 2 + 3 + \cdots + k + (k+1), \quad (4)$$

because the sum of $(k + 1)$ terms equals the sum of k terms, plus the $(k + 1)$st term. By use of (4), the equation 3 which we wish to establish becomes

$$1 + 2 + 3 + \cdots + k + (k + 1) \stackrel{?}{=} \frac{(k + 1)(k + 2)}{2}. \quad (5)$$

To establish (5), add $(k + 1)$ to both sides of (2); we obtain

$$1 + 2 + 3 + \cdots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1). \quad (6)$$

Since
$$\frac{k(k + 1)}{2} + (k + 1) = \frac{k(k + 1) + 2(k + 1)}{2}$$
$$= \frac{(k + 1)(k + 2)}{2},$$

hence, each side of (6) is the same as the corresponding side of (5). Therefore, we have shown that (5) is true if (2) is true, which completes the proof of the auxiliary theorem.

Conclusion. — Let N be any positive integer. Then, to show that (1) is true when $n = N$, we recall that (1) is true, by verification, when $n = 1$, 2, or 3. Now, by use of Part II (where we think of $k = 3$) since (1) is true when $n = 3$, hence (1) is true also when $n = (3 + 1)$ or 4. By use of Part II again, with $k = 4$, it follows that (1) is true when $n = (4 + 1)$ or 5. Proceeding step by step in this manner, by successive applications of Part II, we eventually reach the conclusion that (1) is true when $n = N$. Since N was *any* integer, we have proved Theorem I.

Note. — In order that mathematical induction may be used, it must be possible to arrange the cases, with which the theorem deals,

in a definite order, so that there is a 1st case, a 2d case, ··· a kth case, ···. The total number of cases may be large or small, but the most important type of theorem is one dealing with infinitely many cases.

124. Outline of a proof by mathematical induction:

Part I. *Verification of the truth of the theorem in the first few special cases.*

Part II. *Proof of Auxiliary Theorem. — If the kth case of the theorem is true, then the $(k+1)$st case is also true.*

Conclusion. *The argument for concluding that all cases of the theorem are true.*

NOTE 1. — An argument by mathematical induction may be compared to climbing a ladder, where each round corresponds to a special case of the theorem. In the proof, Part I shows that we are able to climb on to the first round of the ladder. Part II proves that we are able to pass from round to round and, hence, that we are able to arrive eventually at any round, however high.

Example 1. — If n is any positive integer, prove that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \tfrac{1}{3}\, n(n+1)(n+2). \qquad (1)$$

NOTE 2. — There are n terms on the left side of equation 1 and $n(n+1)$ is the formula by which any term can be computed. Thus, the fourth term is $4 \cdot 5$, as obtained by placing $n = 4$ in $n(n+1)$.

Proof; Part I. — Equation 1 is true when $n = 1$, 2, or 3, because from (1) we obtain,

when $n = 1$, $\quad 1 \cdot 2 = \tfrac{1}{3}(1)(1+1)(1+2)$, or $2 = 2$;

when $n = 2$, $\quad 1 \cdot 2 + 2 \cdot 3 = \tfrac{1}{3}(2)(2+1)(2+2)$, or $8 = 8$;

when $n = 3$, $\quad 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 = \tfrac{1}{3}(3)(3+1)(3+2)$, or $20 = 20$.

Part II; Auxiliary Theorem. — *If* (1) *is true when $n = k$, then* (1) *is true also when $n = k + 1$.*

By hypothesis, (1) is true when $n = k$; or

$$1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) = \tfrac{1}{3}\, k(k+1)(k+2). \qquad (2)$$

Under this hypothesis we wish to show the truth of the following equation, which is obtained by placing $n = (k+1)$ in (1):

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (k+1)[(k+1)+1] \overset{?}{=} \tfrac{1}{3}(k+1)[(k+1)+1][(k+1)+2]. \qquad (3)$$

We may write (3) as follows:

$$1\cdot 2+2\cdot 3+\cdots+k(k+1)+(k+1)(k+2) \stackrel{?}{=} \tfrac{1}{3}(k+1)(k+2)(k+3). \qquad (4)$$

To establish (4), add $(k+1)(k+2)$ to both sides of (2); we obtain

$$1\cdot 2+2\cdot 3+\cdots+k(k+1)+(k+1)(k+2) = \tfrac{1}{3}k(k+1)(k+2)+(k+1)(k+2). \qquad (5)$$

When the right member of (5) is simplified it becomes

$$\frac{k(k+1)(k+2)+3(k+1)(k+2)}{3} = \frac{1}{3}(k+1)(k+2)(k+3).$$

Since each side of (5) is the same as the corresponding side of (4), hence we have shown that (3) is true if (2) is true, which completes Part II.

The student should now supply the concluding statement for the proof, as illustrated in the conclusion of the proof of Theorem I.

NOTE 3. — In Part I of a proof by mathematical induction it usually suffices to verify only the first case of the theorem. However, the meaning of the theorem will always be more thoroughly appreciated if several cases are verified.

NOTE 4. — Both Part I and Part II of a proof by mathematical induction are necessary. Thus, Part I by itself would be insufficient because, even though many special cases of a theorem are true, nevertheless the theorem may not be true in all cases. Thus, it can be verified that $(n^2 - n + 41)$ is a prime number for each of the values $n = 1, 2, 3, \cdots, 40$. Hence, it might be inferred that $(n^2 - n + 41)$ is a prime number for all values of the integer n. However, this general result is not true because, on substituting $n = 41$, we find that $(n^2 - n + 41)$ becomes $[(41)^2 - 41 + 41]$, or $(41)^2$, and $(41)^2$ is *not* a prime number.

Likewise, Part II must be accompanied by Part I in order to furnish a proof. Thus, if we should forget the necessity for the verification of special cases in Part I, we could apparently prove the false statement that, *if n is any positive integer, then*

$$1 + 2 + 3 + \cdots + n = 20 + \frac{n}{2}(n+1). \qquad (6)$$

When we compare equation 6 with the correct equation 1, of Section 123, we see that equation 6 is not true for any value of n. Nevertheless, we can easily prove the auxiliary theorem of Part II, which would state that, if

$$1 + 2 + 3 + \cdots + k = 20 + \frac{k(k+1)}{2}, \qquad (7)$$

then, $1 + 2 + 3 + \cdots + (k + 1) = 20 + \dfrac{(k + 1)(k + 2)}{2}$.

This result can be proved by adding $(k + 1)$ to both sides of equation 7.

NOTE 5. — In the natural sciences a general conclusion is often reached, although not demonstrated in the mathematical sense, by a consideration of what happens in a number of special cases. Such reasoning is called *ordinary*, or *incomplete*, induction. In contrast to it, mathematical induction is often called *complete* induction.

EXERCISE 74

By use of mathematical induction prove that each of the following equations is true for all positive integral values of n:

1. $2 + 4 + 6 + \cdots + 2n = n(n + 1)$.

2. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

3. $3 + 6 + 9 + \cdots + 3n = \dfrac{3n}{2}(n + 1)$.

4. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$.

5. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n + 1)} = \dfrac{n}{n + 1}$.

6. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2$.

7. $2 \cdot 4 + 4 \cdot 6 + 6 \cdot 8 + \cdots + 2n(2n + 2) = \frac{1}{3}n(2n + 2)(2n + 4)$.

In Problems 8 to 11 prove the statements by mathematical induction:

8. If n is any positive integer, the sum of the first n multiples of 4 equals $2n(n + 1)$.

9. If n is any positive integer, then $(x^{2n} - y^{2n})$ has $(x + y)$ as a factor.

Proof; Part I. — When $n = 1$, $(x^{2n} - y^{2n})$ becomes $(x^2 - y^2)$, which has $(x + y)$ as a factor. When $n = 2$, $(x^{2n} - y^{2n})$ becomes $(x^4 - y^4)$, which has $(x + y)$ as a factor, because

$$(x^4 - y^4) = (x + y)(x^3 - x^2y + xy^2 - y^3).$$

Part II; Auxiliary Theorem. — *If* $(x^{2n} - y^{2n})$ *has* $(x + y)$ *as a factor when* $n = k$, *then* $(x^{2n} - y^{2n})$ *has* $(x + y)$ *as a factor also when* $n = (k + 1)$.

If $(x^{2n} - y^{2n})$ has $(x + y)$ as a factor when $n = k$, then

$$(x^{2k} - y^{2k}) = (x + y)F, \tag{1}$$

where F is the other factor. When $n = (k + 1)$, $(x^{2n} - y^{2n})$ becomes $(x^{2k+2} - y^{2k+2})$. We verify that *

$$x^{2k+2} - y^{2k+2} = x^2(x^{2k} - y^{2k}) + y^{2k}(x^2 - y^2). \tag{2}$$

Hence, by use of equations 1 and 2, we obtain:

$$\begin{aligned} x^{2k+2} - y^{2k+2} &= x^2(x + y)F + y^{2k}(x + y)(x - y) \\ &= (x + y)\,[x^2F + y^{2k}(x - y)]. \end{aligned}$$

Therefore, $(x^{2k+2} - y^{2k+2})$ has the factor $(x + y)$ if $(x^{2k} - y^{2k})$ has the factor $(x + y)$, and hence the auxiliary theorem has been proved.

The student should supply the concluding statement for the proof.

10. If n is any positive integer, then $(x^n - y^n)$ has $(x - y)$ as a factor.

HINT. — In Part II use the fact that

$$x^{r+1} - y^{r+1} = x(x^r - y^r) + y^r(x - y).$$

11. If n is any positive odd integer, then $(x^n + y^n)$ has $(x + y)$ as a factor.

12. Establish the formula 1 of Section 116 for the nth term of an arithmetic progression, by use of mathematical induction.

13. Establish formula 1 in Section 119 by use of mathematical induction.

* Identity 2 is obtained on dividing $(x^{2k+2} - y^{2k+2})$ by $(x^{2k} - y^{2k})$; the quotient is x^2 and the remainder is $y^{2k}x^2 - y^{2k+2}$, or $y^{2k}(x^2 - y^2)$.

CHAPTER XVII

THE BINOMIAL THEOREM

125. Expansion of $(x + y)^n$. — By multiplication, we obtain the following results:

$$(x + y)^1 = x + y.$$
$$(x + y)^2 = x^2 + 2xy + y^2.$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

We see that, if $n = 1, 2, 3,$ or 4, **the expansion of $(x + y)^n$ contains $(n + 1)$ terms with the following properties:**

I. *In any term the sum of the exponents of x and y is n.*

II. *The first term is x^n, and in each other term the exponent of x is 1 less than in the preceding term.*

III. *The second term is $nx^{n-1}y$, and in each succeeding term the exponent of y is 1 more than in the preceding term.*

IV. *If the coefficient of any term is multiplied by the exponent of x in that term and if the product is divided by the number of that term, the quotient obtained is the coefficient of the next term.*

Thus, in $(x + y)^4$, the third term is $6x^2y^2$. By property IV, we obtain $(6 \cdot 2) \div 3$, or 4, as the coefficient of the fourth term.

V. *The coefficients of terms equidistant from the ends of the expansion are the same.*

Thus, the coefficient of the last term equals that of the first term; the coefficient of the second term equals that of the next to the last term, etc.

At present we shall assume that properties I to V are true for the expansion of $(x + y)^n$, if n is any positive integer

although *we have merely verified their truth when* $n = 1, 2, 3,$ *and* 4. The theorem which justifies this assumption is called the **Binomial Theorem.**

Example 1. — Expand $(c + w)^7$.

Solution. — 1. By use of properties I, II, and III, we obtain

$(c + w)^7 = c^7 + 7\,c^6w + \quad c^5w^2 + \quad c^4w^3 + \quad c^3w^4 + \quad c^2w^5 + \quad cw^6 + w^7$,

where spaces are left for the unknown coefficients.

2. By property IV, the coefficient of the third term is $(7 \cdot 6) \div 2$, or 21; that of the fourth term is $(21 \cdot 3) \div 4$, or 35.

3. By property V, we obtain the other coefficients and hence

$(c + w)^7 = c^7 + 7\,c^6w + 21\,c^5w^2 + 35\,c^4w^3 + 35\,c^3w^4 + 21\,c^2w^5 + 7\,cw^6 + w^7$.

Example 2. — Expand $\left(2\,a - \frac{w}{3}\right)^6$.

Solution. — 1. $\left(2\,a - \frac{w}{3}\right)^6 = \left[(2\,a) + \left(-\frac{w}{3}\right)\right]^6$.

2. We use properties I to V with $x = 2\,a$ and $y = -\frac{w}{3}$:

$$\left(2\,a - \frac{w}{3}\right)^6 = (2\,a)^6 + 6(2\,a)^5\left(-\frac{w}{3}\right) + 15(2\,a)^4\left(-\frac{w}{3}\right)^2 + 20(2\,a)^3\left(-\frac{w}{3}\right)^3$$
$$+ 15(2\,a)^2\left(-\frac{w}{3}\right)^4 + 6(2\,a)\left(-\frac{w}{3}\right)^5 + \left(-\frac{w}{3}\right)^6, \text{ or,}$$

$$\left(2\,a - \frac{w}{3}\right)^6 = 64\,a^6 - 64\,a^5w + \frac{80}{3}\,a^4w^2 - \frac{160}{27}\,a^3w^3 + \frac{20}{27}\,a^2w^4 - \frac{4}{81}\,aw^5 + \frac{w^6}{729}.$$

NOTE. — The signs are alternately plus and minus in the expansion of a power of a binomial whose first term bears a plus sign and second term bears a minus sign.

EXERCISE 75

By use of properties I *to* V, *expand each power:*

1. $(x + y)^5$.
2. $(a + b)^7$.
3. $(a + b^2)^4$.
4. $(m - n)^5$.
5. $(a - 3\,b)^4$.
6. $(-x + 2)^3$.
7. $(r^{\frac{1}{2}} - 1)^6$.
8. $(2\,m^2 - 1)^4$.
9. $(a^2 + x^2y)^4$.
10. $(-a - 2)^5$.
11. $(x^{\frac{1}{4}} + 1)^3$.
12. $(a^{\frac{1}{3}} - x)^7$.

Find the first three terms of the expansion of each power:

13. $(a - 3)^{15}$.
14. $(m^2 + 2n)^{20}$.
15. $\left(\frac{1}{a} + \frac{1}{y}\right)^6$.
16. $(5 - \sqrt{2})^5$.
17. $(a - \frac{1}{2})^{16}$.
18. $(-b^3 + a^3)^{10}$.
19. $\left(\frac{a}{b} + \frac{b}{a}\right)^8$.
20. $(1 + .02)^{10}$.
21. $(x^3 + 3y)^{12}$.
22. $(x + y)^n$.
23. $(x^{\frac{1}{2}} - y^{\frac{1}{2}})^7$.
24. $(a^{-1} + b^{-2})^5$.

Definition. The symbol $n!$ is read *n factorial* and is an abbreviation for the product of all integers from 1 to n inclusive.

Thus, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

Find the value of each symbol:

25. $4!$ **26.** $3!$ **27.** $7!$ **28.** $6!$ **29.** $\frac{5!}{2!}$ **30.** $\frac{9!}{4!}$

126. The general term of $(x + y)^n$. — By use of properties I to IV of Section 125, we obtain

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{1 \cdot 2}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{n-3}y^3 + \cdots. \quad (1)$$

In (1), notice that the following statements are true if $r = 1$, 2, or 3.

Description of the term involving y^r, in $(x + y)^n$:

A. *The exponent of x is $(n - r)$.*

B. *The denominator is the product of all integers from 1 to r inclusive.*

C. *The numerator of the coefficient has r factors, the first being n and each other being 1 less than the preceding factor. The last factor is $(n - r + 1)$.*

When A, B, and C are combined, they state that

the term involving y^r is $\frac{n(n-1)\cdots(n-r+1)}{r!}x^{n-r}y^r$. (2)

It will be proved later that (2) is true for all values of r. For the present, we shall assume this fact without proof.

To obtain a formula for the rth term in the expansion of $(x + y)^n$, we notice that this term will contain y^{r-1} as a factor. Hence, we substitute $(r - 1)$ for r in formula 2 and find that

$$\text{the } r\text{th term is } \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}x^{n-r+1}y^{r-1}. \qquad (3)$$

By use of formulas 2 and 3, we may write

$$\left.\begin{aligned}(x+y)^n = {} & x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots \\ & + \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}x^{n-r+1}y^{r-1} \\ & + \frac{n(n-1)\cdots(n-r+1)}{r!}x^{n-r}y^r + \cdots + y^n.\end{aligned}\right\} \qquad (4)$$

Equation 4 is referred to as the **Binomial Formula.**

NOTE. — Either of the formulas 2 and 3 may be called the *general term* of the expansion of $(x + y)^n$ because either one enables us to compute any particular term of $(x + y)^n$ without writing the other terms.

Example 1. — Find the 8th term of $(3\,a^{\frac{1}{2}} - b)^{11}$.

Solution. — The 8th term will involve the 7th power of the second term of the binomial. Hence, use (2) with $r = 7$, $x = 3\,a^{\frac{1}{2}}$, and $y = -b$:

$$\text{8th term is } \frac{11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}(3\,a^{\frac{1}{2}})^4(-b)^7 = -26,730\,a^2b^7.$$

NOTE. — Example 1 can also be solved by use of formula 3 with $r = 8$.

EXERCISE 76

By use of formula 2 *or formula* 3, *find the following terms:*

1. 4th term of $(a + x)^8$.
2. 9th term of $(x + y)^9$.
3. 9th term of $(m - n)^{11}$.
4. 6th term of $(x^m - y^n)^{12}$.
5. Term in the expansion of $(2x^2 + z^3)^8$ involving z^{12}.
6. Term involving y^{10} in the expansion of $(x - y^2)^7$.
7. Term involving y^5 in the expansion of $(x + y)^n$.

8. 8th term of $(a^3 + x^5)^{10}$.

9. 3d term of $(x^{-1} - 2y^{\frac{1}{2}})^{13}$.

10. 10th term of $(-x + 2a)^{12}$.

11. 7th term of $(-2 + c)^{11}$.

12. Middle term of $(3 - x)^6$.

13. Middle term of $(2x + 5)^8$.

14. Term of $(x + y)^{16}$ which has the largest coefficient.

HINT. — If n is an even integer, the middle term of $(x + y)^n$ has the largest coefficient. If n is an odd integer, the largest coefficient is attached to the two central terms.

15. Terms of $(x + y)^7$ having the largest coefficient.

Solve each problem by use of the Binomial Formula:

16. Find the value of $(98)^5$.

HINT. — Notice that $98 = (100 - 2)$.

17. Find the value of $(1.01)^{10}$ correct to four decimal places.

HINT. — Notice that $(1.01)^{10} = (1 + .01)^{10}$. Use only as many terms as are necessary.

18. Find $(1.03)^7$, correct to three decimal places.

NOTE. — Numerical applications of the Binomial Formula, when the exponent is not an integer, are found in Section 226.

127. Binomial Theorem. — *The Binomial Formula for $(x + y)^n$ is true for every positive integral value of n.*

We shall prove this theorem by mathematical induction.

Proof; Part I. — In previous problems, we have verified the Binomial Formula when $n = 1$, 2, 3, and 4 (as well as when n has other values). Hence, we shall consider that Part I of our proof has been completed.

Part II; Auxiliary Theorem. — *If the Binomial Formula is true when $n = k$, then the formula is true when $n = k + 1$.*

By hypothesis, equation 4, page 181, is true when $n = k$, or

$$(x + y)^k = x^k + kx^{k-1}y + \frac{k(k-1)}{2!}x^{k-2}y^2 + \cdots \tag{1}$$

$$+ \frac{k(k-1)\cdots(k-r+2)}{(r-1)!}x^{k-r+1}y^{r-1} + \frac{k(k-1)\cdots(k-r+1)}{r!}x^{k-r}y^r + \cdots + y^k.$$

Under this assumption we wish to prove that $(x + y)^{k+1}$ is correctly given by the following expression, which we obtain by substituting $n = k + 1$ in the right member of equation 4, page 181:

$$x^{k+1} + (k+1)x^k y + \frac{(k+1)k}{2!} x^{k-1}y^2 + \cdots \tag{2}$$

$$+ \frac{(k+1)(k)\cdots[(k+1)-r+1]}{r!} x^{(k+1)-r}y^r + \cdots + y^{k+1}.$$

Since $(x+y)^{k+1} = (x+y)(x+y)^k$, we can obtain $(x+y)^{k+1}$ by multiplying the right member of (1) by $(x+y)$. On multiplying, the first few terms obtained are as follows:

$$x^{k+1} + kx^k y + \frac{k(k-1)}{2!} x^{k-1}y^2 + \cdots \tag{3}$$

$$+ x^k y + kx^{k-1}y^2 + \cdots$$

In (3), the first line results from use of the x of the multiplier $(x+y)$ and the second line from use of the y. On adding the terms in these two lines, we obtain

$$x^{k+1} + (k+1)x^k y + \frac{k(k+1)}{2!} x^{k-1}y^2 + \cdots,$$

which are the same as the first three terms in (2).

We must also show that, in the result of the multiplication by $(x+y)$, the general term involving y^r is identical with the corresponding term in (2). In multiplying $(x+y)^k$ by $(x+y)$, we obtain terms involving y^r by multiplying

(*a*) the term in y^r in (1) by the x of $(x+y)$, and
(*b*) the term in y^{r-1} in (1) by the y of $(x+y)$.

That is, we obtain, as the terms in y^r,

$$\frac{k(k-1)\cdots(k-r+1)}{1\cdot 2\cdots r} x^{k-r+1}y^r + \frac{k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots(r-1)} x^{k-r+1}y^r. \tag{4}$$

Reducing these to a common denominator and adding, we obtain

$$\frac{[(k-r+1)+r](k)(k-1)\cdots(k-r+2)}{1\cdot 2\cdots r} x^{k-r+1}y^r. \tag{5}$$

Since $(k-r+2) = [(k+1)-r+1]$, hence (5) can be written

$$\frac{(k+1)(k)(k-1)\cdots[(k+1)-r+1]}{r!} x^{(k+1)-r}y^r,$$

which is the same as the corresponding term in (2).

On multiplying y^k in (1) by the y of $(x+y)$, we obtain y^{k+1}, which is the last term in (2). Hence, we have shown that, if (1) is true, then $(x+y)^{k+1}$ is given by (2). Hence, the auxiliary theorem is proved.

It is left to the reader to make the usual concluding statement of a proof by mathematical induction, to complete the proof of the Binomial Theorem.

CHAPTER XVIII

INEQUALITIES

128. Definitions.* — To say that a *is greater than* b, $a > b$, means that $(a - b)$ is positive. To say that c *is less than* d, $c < d$, means that $(c - d)$ is negative. Any relation between numbers expressed by use of $<$ or $>$ is called an **inequality**. An expression $A \leqq B$ is read "A *is less than or equal to* B."

If an inequality does not involve any literal numbers, or if it is true for all values of the letters involved, the inequality is called an **absolute inequality**. A **conditional inequality** is one which is true only for certain values of the letters involved.

Thus, $8 > 3$ is an absolute inequality; $a^2 + d^2 + 3 \geqq 3$ is an absolute inequality because, for all values of a and d, $a^2 + d^2 \geqq 0$. $x - 3 \leqq 0$ is a conditional inequality because it is true only if $x \leqq 3$.

In an inequality $M < N$, we speak of M as the left member, or side, and N as the right member. Two inequalities are said to have the **same sense**, if, in them, the inequality signs point in the same direction. Thus, $A < B$ and $C < D$ have the same sense.

129. Properties of inequalities. — The sense of an inequality $A > B$ is not changed by any of the following operations:

I. *Adding the same number to both sides.*

II. *Multiplying or dividing both sides by the same positive number.*

* The relation $a > b$ is defined only for real numbers a and b, and hence it is *meaningless* to say that one imaginary number is greater than another. All literal numbers in this chapter are real.

Proof of I. — **1.** If C is any real number, we wish to prove that $A + C > B + C$.

2. By definition, since $A > B$, then $A - B = p$, where p is some positive number. Hence, $(A + C) - (B + C) = A - B = p$, or $[(A + C) - (B + C)]$ is positive. Therefore, $A + C > B + C$.

. To subtract a number N, we add $-N$. Hence, from I, it follows that the sense of an inequality is not changed by *subtracting the same number from both sides.*

Proof of II. — **1.** If k is any positive number, we wish to prove that $kA > kB$.

2. Since $A - B = p$, hence $kA - kB = k(A - B) = kp$. Since p is positive, and since k is positive, hence kp is positive. Therefore, $(kA - kB)$ is positive and hence $kA > kB$.

NOTE. — Property I justifies the following mechanical operations on inequalities:

A term appearing on both sides of an inequality may be **cancelled** by subtracting the term from both sides.

A term may be **transposed** *from one member of an inequality to the other, with the sign of the term changed,* by subtracting the term from both members.

An inequality $A > B$ is said to be **linear** in x if A and B are linear in x. By use of properties I and II, we can find the values of x for which a linear inequality is true.

Example **1.** — Find the values of x for which

$$\tfrac{7}{3}x - 1 < 17 - \frac{2x}{3}.$$

Solution. — 1. Multiply both sides by 3: $7x - 3 < 51 - 2x$.

2. Add $(3 + 2x)$ to both sides: $9x < 54$.

3. Divide both sides by 9: $x < 6$.

4. Hence, the inequality is true when and only when $x < 6$.

III. *The sense of an inequality $A > B$ is reversed if both sides are multiplied by* -1.

Proof. — **1.** We wish to prove that $-A<-B$.

2. Since $A>B$, hence $(A-B)$ is positive. Since $[-B-(-A)]=-B+A=A-B$, therefore $[-B-(-A)]$ is positive. Hence, $-B>-A$.

Property III shows that **if all signs are changed in an inequality the inequality sign must be reversed.**

Thus, if $x-2<4$, then $-x+2>-4$.

EXERCISE 77

For what values of x is the inequality true?

1. $3x-7>0$.
2. $5-4x<0$.
3. $\frac{2}{3}x+3>2$.
4. $\frac{3}{4}x+5>3$.
5. $4x-5>-9+3x$.
6. $2x+18<3-3x$.
7. $5-7x>10-10x$.
8. $2x-4>\frac{1}{3}x-2$.
9. $5x-\frac{3}{5}<2-4x$.
10. $3x-\frac{1}{5}<5x+2$.

130. Graphical solution of inequalities in one variable. Any inequality in one variable can be placed in the form $f(x)>0$, by transposing all terms to the "greater than" side of the inequality.

Thus, $-3x-5x^2<3+5x$ becomes $0<5x^2+8x+3$.

To solve an inequality $f(x)>0$ graphically, *draw a graph of $f(x)$. Then, the values of x for which the graph of $f(x)$ is above the x-axis are the values of x for which the inequality $f(x)>0$ is true.*

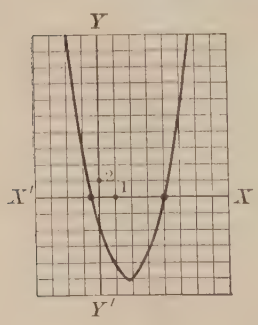


FIG. 12.

A similar method applies to inequalities of the form $f(x)<0$.

Example 1. — Find the values of x for which $2x^2>7x+4$.

Solution. — 1. Transpose all terms to the left side: $2x^2-7x-4>0$.

2. Let $f(x) = 2x^2 - 7x - 4$. A graph of $f(x)$ is given in Figure 12. From the graph it is seen that $f(x) > 0$ if $x > 4$ or if $x < -\frac{1}{2}$.

NOTE. — To solve $f(x) > 0$, it is essential that the graph of $f(x)$ should show accurately the points where $f(x) = 0$. In other respects, the graph need not be very accurate.

EXERCISE 78

Determine graphically the values of the variable for which the inequality holds:

1. $7x - 5 > 0$.
2. $-3x + 2 > 0$.
3. $x^2 + 2x - 8 > 0$.
4. $x^2 - 13x + 40 > 0$.
5. $6 - x < 2x^2$.
6. $2 - 3x^2 < x$.
7. $x^2 < x + 6$.
8. $x^2 + 15 < 8x$.
9. $9a^2 < 16$.
10. $m^2 < 25$.
11. $x^2 + 3x + 8 < 0$.
12. $15a^2 + 10a + 4 > 0$.

For each equation, find the values of k for which the roots are real and not equal, and those values for which the roots are imaginary:

HINT. — In each problem, first compute the discriminant D and then find the values of k for which $D > 0$, and those for which $D < 0$.

13. $3x^2 + 5x - 5k = 0$.
14. $2kx^2 + 4x + 3 = 0$.
15. $x^2 + 3kx + 5 = 0$.
16. $kx^2 - 5x + 2k = 0$.

SUPPLEMENTARY PROBLEMS

17. For what values of k will the straight line, which is the graph of $y = kx - 5$, have no points in common with the circle $x^2 + y^2 = 9$?

HINT. — 1. Substituting $y = kx - 5$ in $x^2 + y^2 = 9$, we obtain

$$x^2 + (kx - 5)^2 = 9, \text{ or } x^2 + k^2x^2 - 10kx + 25 = 9. \qquad (1)$$

2. If the straight line does not intersect the circle, then (1) has imaginary roots. Find the values of k for which (1) has imaginary roots.

18. For what values of k will the straight line $y = kx + 3$ meet the curve, which is the graph of $2x^2 + y^2 = 4$, in two different points?

19. Find the values of m for which the straight line $y = mx - 6$ will meet the circle $x^2 + y^2 = 25$, (*a*) in no points; (*b*) in two distinct points; (*c*) in two coincident points.

20. Prove that, if x and y are not equal, then $x^2 + y^2 > 2xy$.

Proof. — 1. IF $x^2 + y^2 > 2xy$, (1)

then $x^2 - 2xy + y > 0$; (2)

then $(x - y)^2 > 0$. (3)

2. But, $(x - y)^2 > 0$ is true if $x \neq y$; hence (2) is true; hence, by adding $2xy$ to both sides of (2), we obtain (1). Therefore, (1) is true.

NOTE. — In Problem 20, we assumed that the desired inequality 1 was true and then derived successive inequalities from it. We eventually reached the obviously true inequality 3. Then, starting from this true inequality, we reasoned that we could derive (1), and hence that (1) is true. This type of proof can be used in the following problems.

21. Prove that, if $x>0$, $y>0$, and $x \neq y$, then $\frac{x+y}{2} > \frac{2xy}{x+y}$.

22. Prove that, if $x>0$, $y>0$, and $x \neq y$, then $\frac{x+y}{2} > \sqrt{x}\sqrt{y}$.

NOTE. — Problem 22 states that the *arithmetic mean* of x and y is greater than the *geometric mean*, $\sqrt{xy}$.

23. Prove that, if $a > 0$ and $a \neq 1$, then $a + \frac{1}{a} > 2$.

24. Prove that, if $x > 0$ and $y > 0$, then $\frac{x}{y} + \frac{y}{x} > -2$.

Definition. *The symbol "$|N|$" is used to denote the* **absolute value** *of N.*

Thus, $|5|=5$; $|-37|=37$. $|-\frac{3}{4}|=\frac{3}{4}$.

25. If x and y are any real numbers, prove that

$$|x+y| \leqq |x|+|y|.$$

26. If A is positive, show that the inequality $x^2 \leqq A$ is satisfied if $|x| \leqq \sqrt{A}$.

CHAPTER XIX

COMPLEX NUMBERS

131. Complex numbers. — If we let $i = \sqrt{-1}$, and if a and b are real,* then $(a + bi)$ is called a **complex number**, whose **real part** is a and **imaginary part** is bi; we call b the *coefficient of the imaginary part.* Any real number a is considered as a complex number, in which the coefficient of the imaginary part is zero.

Thus, $a = a + 0i$; $6 = 6 + 0i$; $0 = 0 + 0i$.

A complex number $(a + bi)$ is called an **imaginary number** if $b \neq 0$. If $a = 0$ and $b \neq 0$, then $(a + bi)$ is called a **pure imaginary number.**

Thus, $(3 + 5i)$ is an imaginary number; $6i$ is a pure imaginary number.

Definition I. — *Two complex numbers* $(a + bi)$ *and* $(c + di)$ *will be called equal, in case* $a = c$ *and* $b = d$.

Since $0 = 0 + 0i$, it follows from Definition **I** that

$$\textbf{if } a + bi = 0, \textbf{ then } a = 0 \textbf{ and } b = 0. \qquad (1)$$

132. Algebraic operations with complex numbers. — By definition, $i^2 = -1$; $i^3 = i \cdot i \cdot i = i \cdot i^2 = i(-1)$, or $i^3 = -i$; $i^4 = i^2 i^2 = (-1)(-1)$ or $i^4 = 1$; $i^5 = i \cdot i^4 = i \cdot 1$, or $i^5 = i$; etc., any positive integral power of i is either $+1$, -1, $+i$, or $-i$.

* In this chapter, i will always represent $\sqrt{-1}$ and all other literal numbers will represent real numbers. Review Section 83.

We agree that,* in any indicated sum, difference, product, or quotient involving integral powers of complex numbers, the result shall be that which is obtained by operating as if i were a literal number subject to the rules of algebra as they apply to real numbers. In the final result of any operation, and at the intermediate stages, each positive integral power of i should be expressed properly as i, $-i$, 1, or -1, as in the preceding paragraph.

Illustrations. $(3 + 5i) - (5 - 6i) = (3 - 5) + (5i + 6i) = -2 + 11i.$

$(3 + i)(5 + 2i) = 15 + 11i + 2i^2 = 15 + 11i - 2 = 13 + 11i.$

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

$$k(a + bi) = ka + kbi.$$

$$(a + bi)(c + di) = ac + (bc + ad)i + bdi^2, \text{ or}$$

$$(a + bi)(c + di) = (ac - bd) + i(bc + ad).$$

133. Conjugate complex numbers. If two complex numbers differ only in the sign of their imaginary parts, the two numbers are called **conjugate complex numbers**, and either is called the **conjugate** of the other.

Thus, the conjugate of $(a + bi)$ is $(a - bi)$. Moreover,

$$(a + bi) - (a - bi) = 2bi; \tag{1}$$

$$(a + bi) + (a - bi) = 2a; \tag{2}$$

$$(a + bi)(a - bi) = a^2 - b^2i^2 = a^2 + b^2. \tag{3}$$

From (1), (2), and (3), it follows that the difference of two conjugate complex numbers is a pure imaginary number whereas their sum and their product are real numbers.

A quotient of two complex numbers can be reduced to the form $a + bi$ *by multiplying both numerator and denominator by the conjugate of the denominator.*

* For a complete logical foundation of the algebra of complex numbers, see Dickson's ELEMENTARY THEORY OF EQUATIONS, page 1.

Thus, $\frac{5+2i}{3-4i} = \frac{5+2i}{3-4i} \cdot \frac{3+4i}{3+4i} = \frac{15+26i+8i^2}{9-16i^2} = \frac{7}{25} + \frac{26i}{25}$.

Hence, $\frac{5+2i}{3-4i} = \frac{7}{25} + \frac{26}{25}i$.

EXERCISE 79

In an indicated square root, whenever the radicand has a minus sign attached, the radical should be simplified and expressed in terms of i, as in Section 83.

Simplify and express by use of i:

1. $\sqrt{-7}$. **3.** $\sqrt{-25}$. **5.** $\sqrt{-98}$.

2. $\sqrt{-4x^2}$. **4.** $\sqrt{-\frac{18}{7}}$. **6.** $\sqrt{-200}$.

Find the value of each power of i:

7. i^6. **8.** i^7. **9.** i^{10}. **10.** i^{13}. **11.** i^9. **12.** i^8.

Perform indicated operations and simplify to the form a + bi:

13. $(5+3i)-(2+6i)$. **15.** $2-\sqrt{-4}-(3-5\sqrt{-4})$.

14. $(3-2i)+(4-7i)$. **16.** $6+3\sqrt{-5}+2-5\sqrt{-5}$.

17. $3i(6i^2)$. **19.** $(5\sqrt{-3})^2$. **21.** $2\sqrt{-3}\sqrt{-5}$. **23.** $(2i)^5$.

18. $5i(-4i)$. **20.** $(6\sqrt{-12})^2$. **22.** $3\sqrt{2}\sqrt{-2}$. **24.** $(4i)^3$.

25. $(3-5i)(6+7i)$. **27.** $(4+3\sqrt{-2})(4-3\sqrt{-2})$.

26. $(2-4i)(2+4i)$. **28.** $(5+3\sqrt{-2})(6-\sqrt{-18})$.

Tell the conjugate of each complex number:

29. $5+6i$. **31.** $3\sqrt{-7}+6$. **33.** $6i$.

30. $3-4i$. **32.** $-5i+2$. **34.** $\sqrt{-4}-5$.

Reduce to standard form a + bi:

35. $\frac{5+3i}{2+4i}$. **36.** $\frac{3+5i}{-3i+4}$. **37.** $\frac{-5}{7i-2}$.

38. $\dfrac{2+\sqrt{-2}}{3-5\sqrt{-2}}$. **41.** $\dfrac{3}{2i}$. **44.** $(3\sqrt{-2})^5$.

39. $\dfrac{5\sqrt{3}}{3+i\sqrt{3}}$. **42.** $-\dfrac{1}{i}$. **45.** $(2i+5)^{-1}$.

40. $\dfrac{1}{3+7i}$. **43.** $(2i+5)^3$. **46.** $(i+2)^{-2}$.

47. Find the reciprocal of each complex number:

$$3+5i;\ 7i;\ 6+\sqrt{-7}.$$

48. Prove that, if $a^2+b^2=1$, the reciprocal of $a+bi$ is $a-bi$.

By use of Definition I *and statement* 1, *of Section* 131, *find* x *and* y *under the given conditions:*

49. $x+yi=3+2i$. **51.** $(2x+3)+(2x-y)i=0$.

50. $(x+3)+(y-4)i=0$. **52.** $x-3y+(x+2y-6)i=0$.

SUPPLEMENTARY MATERIAL

134. Geometric representation of complex numbers. In Figure 13, we shall call $X'X$ the axis of real numbers, and $Y'Y$ the axis of pure imaginary numbers. Then, we agree to represent any complex number $(a+bi)$ geometrically by the point whose x-coördinate is a and y-coördinate is b.

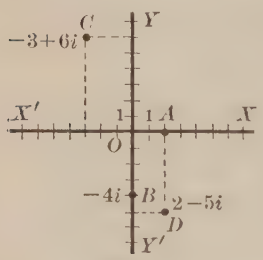


FIG. 13.

Thus, in Figure 13, D represents $(2-5i)$; C represents $(-3+6i)$; A represents the real number 2, which is thought of as $(2+0i)$; B represents the pure imaginary number $-4i$, which is thought of as $(0-4i)$.

The imaginary part of a real number is zero; $a=a+0i$. Hence, all real numbers are represented geometrically by the

points on the axis of real numbers in Figure 13. Similarly, all pure imaginary numbers are represented by the points on $Y'Y$, because the real part of a pure imaginary number is 0.

EXERCISE 80

Represent each complex number geometrically:

1. $3 + 2i$. **3.** -8. **5.** $3i$. **7.** $5 - i\sqrt{2}$.

2. $5 - 8i$. **4.** 4. **6.** $-6i$. **8.** $\frac{1}{3} + \frac{5}{2}i$.

9. $-5 - 6i$. **10.** $-5 + 6i$.

11. Represent geometrically $(5 - 6i)$ and its conjugate.

135. Geometric construction of the sum of two complex numbers:

I. *Let points M and N* (in Fig. 14) *represent $(a + bi)$ and $(c + di)$, respectively. Connect the origin O with M and N.*

II. *Complete the parallelogram, having OM and ON as adjacent sides. Then, the fourth vertex P represents the sum of $(a + bi)$ and $(c + di)$.*

Proof. — **1.** In Figure 14,

$$OR = OS + SR = OS + MH.$$

But, since triangles OKN and MHP are equal, $MH = OK$. Hence,

$$OR = OS + OK = a + c.$$

Similarly, $RP = SM + KN = b + d$.

2. Hence, the coördinates of P are $[(a + c), (b + d)]$, and therefore P represents $(a + c) + (b + d)i$, which is the sum of $(a + bi)$ and $(c + di)$.

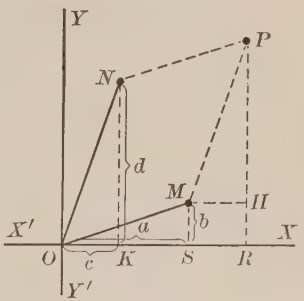


FIG. 14.

Since $(a + bi) - (c + di) = (a + bi) + (-c - di)$, therefore, *to subtract* $(c + di)$ from $(a + bi)$ geometrically, we geometrically add $(-c - di)$ to $(a + bi)$.

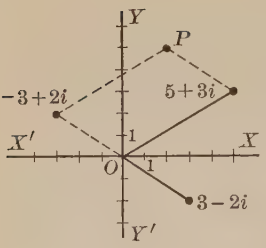


FIG. 15.

Thus, in Figure 15 we have found the point P representing

$$(5 + 3i) - (3 - 2i)$$

by adding $(5 + 3i)$ and $(-3 + 2i)$.

EXERCISE 81

Construct each sum or difference graphically; then, read the value of the sum or difference from the figure:

1. $(3 + i)+(2 + 3i)$.
2. $(2 - 3i)+(1 - 2i)$.
3. $(2 + 4i)-(2i - 5)$.
4. $(-6 + 2i)-(-7i - 4)$.
5. $3 +(5i - 7)$.
6. $5i +(3 - 2i)$.
7. $(7 + 0i)+(0 + 8i)$.
8. $(12i)+(8)$.
9. $(3 + 8i)-(2i - 3)$.
10. $(8i + 3)-(4i - 8)$.
11. $3i -(5i - 6)$.
12. $6 -(4 + 2i)$.

13. Graph each of the numbers and then find the sum of the three geometrically: $(3 + 2i)$; $(5 - 3i)$; $(6 + 2i)$.

14. Add geometrically: $(3 + 5i)+(2 - 6i)+(-3 + 4i)$.

136. Trigonometric form of complex numbers.* — In Figure 16, P represents $(a + bi)$, and the units of length on the two axes are the same. Let r be the length of OP; let θ be the angle XOP, measured either clockwise or counter-clockwise from the line OX. Then,

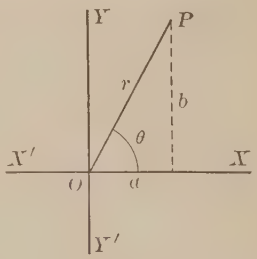


FIG. 16.

$$r = \sqrt{a^2 + b^2}; \quad \tan\theta = \frac{b}{a}; \tag{1}$$

$$a = r\cos\theta; \quad b = r\sin\theta; \tag{2}$$

$$a + bi = r(\cos\theta + i\sin\theta). \tag{3}$$

* A knowledge of trigonometry is presupposed in the rest of this chapter.

We call $r(\cos\theta + i\sin\theta)$ the **trigonometric** (or, **polar**) form of $(a + bi)$. We call θ the **amplitude** (or, **argument**), and the positive length r the **modulus** (or, **absolute value**) of $(a + bi)$.

The amplitude of $(a + bi)$ may be taken as *any* angle whose initial side is OX and whose terminal side is OP, because all such angles have the same trigonometric functions. Hence, if θ is one amplitude, the other permissible amplitudes are $(\theta + k \cdot 360°)$, where k takes all integral values.

Thus, if $\theta = 20°$ is one amplitude, we may also select any of the following as amplitudes: 380°, 740°, − 340°, − 700°, etc.

In writing a complex number in polar form we usually select the smallest permissible positive angle as the amplitude. If two complex numbers in polar form are equal, their moduli are equal, and their amplitudes are either equal or else *differ by some integral multiple of* 360°.

The number zero may be written $0 = 0 \cdot (\cos\theta + i\sin\theta)$ where θ has any value. That is, the modulus of zero is 0, and the amplitude is *any* angle, whatever.

To express a given number $(a + bi)$ in its trigonometric form, first find r from $r = \sqrt{a^2 + b^2}$. Then, notice what quadrant θ is in, by plotting $(a + bi)$, and find the value of θ by use of $\tan\theta = \dfrac{b}{a}$, with the aid of a table of trigonometric tangents.

Example 1. — Find the trigonometric form of $(5 - 6i)$.

Solution. — 1. $r = \sqrt{61}$, and θ is in the 4th quadrant.
2. Tan $\theta = -\frac{6}{5} = -1.200$; from Table II, $\theta = 309.8°$.

Example 2. — Reduce to the form $(a + bi)$:
$$6(\cos 135° + i\sin 135°).$$

Solution. — From Table II, $\cos 135° = -.707$; $\sin 135° = .707$.
Hence, $6(\cos 135° + i\sin 135°) = -4.242 + 4.242i$.

To plot $r(\cos\theta + i\sin\theta)$, draw a circle on the coördinate system with O as center and r as radius; measure angle XOP equal to θ. The point where the terminal side of θ cuts the circle is the point representing $r(\cos\theta + i\sin\theta)$.

EXERCISE 82

Plot each number:

1. $5(\cos 45° + i\sin 45°)$.
2. $3(\cos 240° + i\sin 240°)$.
3. $2[\cos(-60°) + i\sin(-60°)]$.
4. $8(\cos 0° + i\sin 0°)$.
5. $6(\cos 180° + i\sin 180°)$.
6. $4(\cos 270° + i\sin 270°)$.
7. $3(\cos 90° + i\sin 90°)$.
8. $5(\cos 225° + i\sin 225°)$.
9. $3(\cos 72° + i\sin 72°)$.
10. $2(\cos 140° + i\sin 140°)$.

Reduce each number to its trigonometric form:

11. $3 + 3i$.
12. $5 - 5i$.
13. $-4 + 7i$.
14. $0 + 6i$.
15. $5 + 0i$.
16. $-3i$.
17. $4 - 5i$.
18. $-6 - 7i$.
19. $-3 - 4i$.
20. $-\frac{1}{2} + \frac{i}{2}\sqrt{3}$.
21. $-\frac{1}{2} - \frac{i}{2}\sqrt{3}$.

22. Write each of the numbers of problems 1 to 10 in the form $(a + bi)$.

23. Reduce $(1 + i)$ and its conjugate $(1 - i)$ to the trigonometric form.

24. What is the amplitude of any positive real number; of any negative real number; of any pure imaginary number bi, if $b > 0$; of bi if $b < 0$?

25. Prove that the conjugate of $r(\cos\theta + i\sin\theta)$ is $r[\cos(-\theta) + i\sin(-\theta)]$.

137. Product of two numbers in polar form. — *The modulus of the product of two complex numbers in the product of their moduli and the amplitude of the product is the sum of their amplitudes.*

Proof. — We verify that

$r_1(\cos A + i \sin A) \cdot r_2(\cos B + i \sin B)$
$= r_1 r_2(\cos A \cos B + i \sin A \cos B + i \cos A \sin B - \sin A \sin B)$
$= r_1 r_2[(\cos A \cos B - \sin A \sin B) + i(\sin A \cos B + \cos A \sin B)].$

From trigonometry, $(\cos A \cos B - \sin A \sin B) = \cos(A + B)$, and $(\sin A \cos B + \cos A \sin B) = \sin(A + B)$. Hence,

$$r_1(\cos A + i \sin A) \cdot r_2(\cos B + i \sin B) = r_1 r_2[\cos(A + B) + i \sin(A + B)] \quad (1)$$

Thus, $3(\cos 40° + i \sin 40°) \cdot 5(\cos 170° + i \sin 170°) = 15(\cos 210° + i \sin 210°)$.

138. De Moivre's Theorem. — *If n is any positive integer,*

$$[r(\cos A + i \sin A)]^n = r^n(\cos nA + i \sin nA). \quad (1)$$

Proof. — It is obvious that (1) is true when $n = 1$. By use of formula 1 of Section 137, with $r_1 = r$, $r_2 = r$, and $B = A$, we obtain

$$[r(\cos A + i \sin A)]^2 = r^2[\cos(A + A) + i \sin(A + A)] = r^2(\cos 2A + i \sin 2A). \quad (2)$$

Hence, (1) is true also when $n = 2$. Similarly,

$[r(\cos A + i \sin A)]^3 = r(\cos A + i \sin A)[r(\cos A + i \sin A)]^2$
$= r(\cos A + i \sin A) \cdot r^2(\cos 2A + i \sin 2A)$ [From (2)]
$= r^3[\cos(A + 2A) + i \sin(A + 2A)] = r^3(\cos 3A + i \sin 3A).$

As an exercise, the student should complete the proof of (1) by mathematical induction.

Illustration. By De Moivre's Theorem,

$$[3(\cos 20° + i \sin 20°)]^6 = 3^6(\cos 120° + i \sin 120°).$$

139. The nth roots of a complex number.

Example 1. — Find the cube roots of $8(\cos 150° + i \sin 150°)$.

Solution. — 1. Let $r(\cos A + i \sin A)$ be any cube root. Then,

$$8(\cos 150° + i \sin 150°) = [r(\cos A + i \sin A)]^3,$$

or,
$$8(\cos 150° + i \sin 150°) = r^3(\cos 3A + i \sin 3A), \quad (1)$$

where the last expression was obtained by De Moivre's Theorem.

2. It follows from (1) that the values of r and A, which give cube roots, satisfy the following equations:

$r^3 = 8$, or $r = 2$; $3A = 150° + k \cdot 360°$, or $A = 50° + k \cdot 120°$, (2)

where k is any integer. Placing $k = 0$, 1, and 2 in (2), we obtain, respectively, $A = 50°$, $A = 170°$, and $A = 290°$. These give the following cube roots:

$2(\cos 50° + i \sin 50°)$; $2(\cos 170° + i \sin 170°)$; $2(\cos 290° + i \sin 290°)$.

3. When $k = 3$ in (2), $A = 50° + 360°$, which is equivalent to the amplitude 50°. If $k = -1$, $A = 50° - 120° = -70° = 290° - 360°$, which is equivalent to 290°. Similarly, if k has *any* integral value in (2), the value of A obtained is equivalent to one or other of (50°, 170°, 290°). Hence, the roots obtained in Step 2 are the only cube roots of

$$8(\cos 150° + i \sin 150°).$$

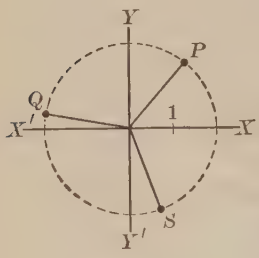


FIG. 17.

NOTE. — The cube roots found in Example 1 are represented by P, Q, and S in Figure 17. Notice that these points lie on a circle whose radius is 2, because 2 is the modulus of each of the roots. Moreover, P, Q, and S divide the circumference of this circle into three equal parts because the amplitudes of the roots are (50°, 170°, 290°), where adjacent amplitudes differ by 120°.

NOTE. — In this section, n will always represent a positive integer.

Theorem I. — *Every complex number* $R(\cos B + i \sin B)$, *where* $R > 0$, *has n distinct nth roots.*

Proof. — **1.** Suppose that B is chosen so that $0 \leqq B < 360°$, and let $r(\cos A + i \sin A)$ be any nth root. Then,

$R(\cos B + i \sin B) = [r(\cos A + i \sin A)]^n = r^n(\cos nA + i \sin nA)$, (3)

where the last expression is obtained by use of De Moivre's Theorem.

2. From (3), $r^n = R$, or $r = \sqrt[n]{R}$.

3. From (3), $nA = B + k \cdot 360°$, or $A = \dfrac{B}{n} + k \cdot \dfrac{360°}{n}$, (4)

where k is any integer. On placing $k = 0, 1, 2, \cdots, (n-1)$ in (4), we obtain the following n distinct values of A, all between 0° and 360°:

$$\frac{B}{n};\ \left(\frac{B}{n} + \frac{360°}{n}\right);\ \left(\frac{B}{n} + 2 \cdot \frac{360°}{n}\right);\ \cdots;\ \left(\frac{B}{n} + (n-1)\frac{360°}{n}\right). \quad (5)$$

Corresponding to the amplitudes in (5), we obtain the following n distinct nth roots:

$$\sqrt[n]{R}\left(\cos\frac{B}{n}+i\sin\frac{B}{n}\right),\ \sqrt[n]{R}\left[\cos\left(\frac{B}{n}+\frac{360^\circ}{n}\right)+i\sin\left(\frac{B}{n}+\frac{360^\circ}{n}\right)\right],\ \text{etc.}$$

4. In (4), if k is given any value other than 0, 1, $\cdots$, $(n-1)$, we obtain a value of A differing from one of the amplitudes in (5) by an integral multiple of 360°. Hence, the amplitudes in (5) are the only distinct amplitudes obtained from (4). Therefore, $R(\cos B + i \sin B)$ has exactly n distinct nth roots, as given in Step 3.

Summary. **To obtain the nth roots of $R(\cos B + i \sin B)$,** *place $k = 0, 1, 2, \cdots, n-1$, in the following formula:*

$$\sqrt[n]{R}\left[\cos\left(\frac{B}{n}+k\cdot\frac{360^\circ}{n}\right)+i\sin\left(\frac{B}{n}+k\cdot\frac{360^\circ}{n}\right)\right]. \qquad (6)$$

NOTE. — All nth roots of $R(\cos B + i \sin B)$ have the same modulus, $\sqrt[n]{R}$, and hence all of these nth roots will be located geometrically on the circumference of a circle whose center is the origin and whose radius is $\sqrt[n]{R}$. The points representing the roots divide the circumference into n equal parts, because adjacent amplitudes in (5) differ by $\frac{360^\circ}{n}$. A special case of these remarks is illustrated in Figure 17, page 198.

EXERCISE 83

By use of De Moivre's Theorem, find indicated powers:

1. $[3(\cos 135^\circ + i \sin 135^\circ)]^4$.
2. $[2(\cos 330^\circ + i \sin 330^\circ)]^3$.
3. $(2 + 2i)^5$.
4. $[5(\cos 40^\circ + i \sin 40^\circ)]^4$.
5. $(3 + i)^4$.
6. $(3 - 3i)^3$.

HINT for Problem 3. — First change $(2 + 2i)$ to polar form.

In the rest of this exercise, leave the results in trigonometric form unless otherwise directed.

By the method of Example 1, *page* 197, *find the indicated roots and plot the results, as in Figure* 17:

7. 5th roots of $32(\cos 60^\circ + i \sin 60^\circ)$.
8. 4th roots of $16(\cos 80^\circ + i \sin 80^\circ)$.

9. Cube roots of 1. **10.** 4th roots of -81.

HINT for Problem 9. — Recall that $1 = 1(\cos 0° + i \sin 0°)$.

11. Square roots of $16i$. **12.** 4th roots of $(\sqrt{2} - i\sqrt{2})$.

By use of formula 6 *find the specified roots:*

13. The 4th roots of $64(\cos 132° + i \sin 132°)$.

14. The 5th roots of $243(\cos 80° + i \sin 80°)$.

15. The cube roots of $(3 - i\sqrt{3})$. **18.** The 8th roots of 1.

16. The 4th roots of $(-2 - 2i)$. **19.** The 4th roots of -81.

17. The square roots of i. **20.** The cube roots of -125.

Find all roots of each equation in the form $a + bi$, *and plot them:*

21. $x^4 = 81$. **22.** $x^3 + 1 = 0$. **23.** $x^5 = 243$. **24.** $x^9 - 1 = 0$.

HINT. — The roots of $x^4 = 81$ are, by definition, the 4th roots of 81. Express 81 in trigonometric form and find all 4th roots.

25. Prove that, if M is any complex number and if w represents $(\cos 120° + i \sin 120°)$, which is one of the cube roots of 1, then the cube roots of M are $\sqrt[3]{M}$, $w\sqrt[3]{M}$, and $w^2\sqrt[3]{M}$, where $\sqrt[3]{M}$ is any cube root of M.

140. Division of complex numbers in polar form. — *The modulus of the quotient of two complex numbers is the quotient of their moduli, and the amplitude of the quotient is the amplitude of the numerator minus the amplitude of the denominator.*

Proof. — We verify that

$$\frac{R(\cos B + i \sin B)}{r(\cos A + i \sin A)} = \frac{R}{r} \cdot \frac{(\cos B + i \sin B)(\cos A - i \sin A)}{(\cos A + i \sin A)(\cos A - i \sin A)}$$

$$= \frac{R}{r} \frac{(\cos B + i \sin B)[\cos(-A) + i \sin(-A)]}{\cos^2 A + \sin^2 A}. \qquad (1)$$

In the numerator of the last expression we used the fact that

$$\cos A = \cos(-A) \text{ and } \sin(-A) = -\sin A.$$

From Section 137, it follows that

$$(\cos B + i \sin B)[\cos(-A) + i \sin(-A)]$$
$$= \cos(B - A) + i \sin(B - A). \quad (2)$$

From trigonometry, $\cos^2 A + \sin^2 A = 1$. Hence, from (1) and (2),

$$\frac{R(\cos B + i \sin B)}{r(\cos A + i \sin A)} = \frac{R}{r}[\cos(B - A) + i \sin(B - A)]. \quad (3)$$

EXERCISE 84

Find the quotient in trigonometric form:

1. $\dfrac{5(\cos 75° + i \sin 75°)}{3(\cos 150° + i \sin 150°)}$.

2. $\dfrac{12(\cos 100° + i \sin 100°)}{3(\cos 40° + i \sin 40°)}$.

3. $\dfrac{6(\cos 230° + i \sin 230°)}{2 + 2i}$.

4. $\dfrac{3 + 5i}{3(\cos 36° + i \sin 36°)}$.

CHAPTER XX

THEORY OF EQUATIONS

141. An integral rational equation is one in which each term is integral and rational. Every integral rational equation can be put in the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \qquad (1)$$

where n is a positive integer and a_0, a_1, $\cdots$, a_{n-1}, a_n are constants, of which $a_0 \neq 0$. The expression

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \qquad (2)$$

is called a **polynomial**, or an **integral rational function** of degree n in x. The main purpose of this chapter is to study theorems concerning roots of integral rational equations and to develop methods for finding the real roots of any given equation of this type.

NOTE. — In this chapter, a functional symbol (see Section 93) $f(x)$, or $H(x)$, etc. will always represent a polynomial in x. The word *equation* will always refer to an integral rational equation.

NOTE. — Polynomials of degrees 1, 2, 3, and 4 are called **linear**, **quadratic**, **cubic**, and **quartic** functions, respectively. Equations of degrees 1, 2, 3, and 4 are called linear, quadratic, cubic, and quartic equations, respectively. Thus, $3x^4 = 5x^2 - 2x + 3$ is a quartic equation.

142. Remainder Theorem. — *If a polynomial $f(x)$ is divided by $(x - r)$ until a remainder independent of x is obtained, this remainder equals $f(r)$.*

Illustration. If $(3x^2 - x + 2)$ is divided by $(x - 2)$, the remainder will be the value of $(3x^2 - x + 2)$ when $x = 2$, or 12.

Proof. — **1.** After $f(x)$ has been divided by $(x - r)$ as described in the theorem, let $q(x)$ be the quotient and let R be the constant remainder. By definition,

(*dividend*) $\equiv$ (*quotient*) $\cdot$ (*divisor*) $+$ (*remainder*). Hence,

$$f(x) \equiv (x - r)q(x) + R. \qquad (1)$$

2. Place $x = r$ in (1): $f(r) = 0 \cdot q(r) + R$; hence, $R = f(r)$.

143. Factor Theorem. — *If* $f(r) = 0$, *then* $(x - r)$ *is a factor of* $f(x)$. In other words, *if* r *is a root of* $f(x) = 0$, *then* $(x - r)$ *is a factor of* $f(x)$.

Proof. — In equation 1 of Section 142, $R = f(r)$. Hence, under our present hypothesis, $R = 0$ and $f(x) \equiv (x - r)q(x)$.

EXERCISE 85

1. If $f(x) = x^3 - 3x + 5$, find $f(3)$; $f(-2)$; $f(a)$; $f(c^2)$.

Thus, $f(3) = 3^3 - 3 \cdot 3 + 5 =$ etc.

2. (*a*) By long division, divide $(x^3 - 3x^2 - 4)$ by $(x + 5)$ until the remainder is a constant. (*b*) Find the remainder by use of the remainder theorem.

By the remainder theorem find the remainders if $(x^4 + 2x^2 - x - 4)$ *is divided by each of the following:*

3. $x - 3$. **5.** $x + 1$. **7.** $x + a$.

4. $x + 2$. **6.** $x - a$. **8.** $x - d^2$.

By use of the factor theorem, answer the following questions:

9. Is $(x - 1)$ a factor of $x^3 - 6x^2 + 11x - 6$?

HINT. — Compute the value of the polynomial when $x = 1$.

10. Is $(x - 4)$ a factor of $x^3 - 6x^2 + 11x - 6$?

11. Is $(x + 2)$ a factor of $x^3 + 3x^2 - 4x - 12$?

12. Is $(x + 4)$ a factor of $x^2 + 3x^2 - 4x - 12$?

13. Is $(x + y)$ a factor of $(x^7 + y^7)$; of $(x^7 - y^7)$?

14. Is $(x - a)$ a factor of $x^5 - a^5$?

Find the values of m for which $(x - 2)$ is a factor of $f(x)$:

15. $f(x) = x^2 - 3mx - 4$. **16.** $f(x) = 3x^3 - 2x^2 + mx - 5$.

Find the values of k for which $(x + 4)$ is a factor of $f(x)$:

17. $f(x) = kx^3 - 3x - 5$. **18.** $f(x) = x^4 - 5kx^2 - 3$.

If n is a positive integer, prove the following statements:

19. $(x^n - 2^n)$ has $(x - 2)$ as a factor.

20. $(x^n - y^n)$ has $(x - y)$ as a factor.

21. $(x^n + y^n)$ has $(x + y)$ as a factor, if n is odd.

22. $(x^n - y^n)$ has $(x + y)$ as a factor, if n is even.

23. Prove the converse to the factor theorem; in other words, prove that, *if $(x - r)$ is a factor of $f(x)$, then r is a root of $f(x) = 0$.*

144. Synthetic division is a process for abbreviating the division of a polynomial $f(x)$ by a binomial $(x - r)$.

Illustration. Consider dividing $(5x^3 - 11x^2 - 14x - 10)$ by $(x - 3)$. The following form I is the ordinary process, and the abridged form II is obtained by omitting non-essential terms from I:

I

$$
\begin{array}{l}
5x^2 + 4x - 2 = \text{quotient.} \\
\hline
5x^3 - 11x^2 - 14x - 10 \;\big|\; x - 3 \\
5x^3 - 15x^2 \\
\hline
\qquad 4x^2 - 14x \\
\qquad 4x^2 - 12x \\
\hline
\qquad\quad -2x - 10 \\
\qquad\quad -2x + 6 \\
\hline
\text{Remainder} = -16
\end{array}
$$

II

$$
\begin{array}{l}
5x^2 + 4x - 2 \\
\hline
5x^3 - 11x^2 - 14x - 10 \;\big|\; x - 3 \\
\qquad -15x^2 \\
\hline
\qquad\quad 4x^2 \\
\qquad\quad -12x \\
\hline
\qquad\qquad -2x \\
\qquad\qquad +6 \\
\hline
\qquad\qquad -16
\end{array}
$$

In $(x - 3)$, the coefficient of x is 1 and hence, at each stage in the division, the coefficient of the highest power of x in the remainder is the next coefficient in the quotient.

In the following form III, we abbreviate II by first writing only the coefficient, in place of each term, and by then condensing all details

into three lines; the quotient is dropped entirely because we know that its coefficients are the heavy black numbers 5, 4, and -2, which we retain.

III:

5	-11	-14	-10	$\lfloor 1 \quad -3$
	-15	-12	$+6$	
5	$+4$	-2	-16	

In III, 5 is introduced into the third row so that all coefficients of the quotient will be seen in one line.

The appearance of III suggests the following form IV, which illustrates the method of synthetic division. In IV, to obtain a number in the second row, we *multiply the preceding heavy black number by* $+3$: we place "$+3$" at the right to recall that this is the multiplier. Each number of the second row is *added to* the number above it to obtain the next number in the third row.

IV:

5	-11	-14	-10	$\lfloor +3$
	$+15$	$+12$	-6	
5	$+4$	-2	-16	

Quotient $= 5x^2 + 4x - 2$. *Remainder* $= -16$.

Rule 1. — To divide $f(x)$ by $(x - r)$ by synthetic division, *arrange $f(x)$ in descending powers of x, supplying each missing power with a zero as its coefficient. Then, arrange the following details in three lines:*

1. *In the first line, write the coefficients a_0, a_1, a_2, $\cdots$, a_n of $f(x)$ in the order in which they appear in $f(x)$.*

2. *Write a_0 in the first place in the third line.*

3. *Multiply a_0 by r, add the product ra_0 to a_1, and write the sum in the third line; multiply this sum by r, add the product to the next coefficient, a_2, and write the sum in the third line; etc., until finally a product is added to the last coefficient of $f(x)$.*

4. *The last number in the third line is the remainder and the other numbers in the third line are the coefficients of the powers of x in the quotient, arranged in descending powers of x.*

Example 1. — Divide $(2x^4 - 12x^2 - 5)$ by $(x + 3)$.

Solution. —

2	0	-12	0	-5	-3
	-6	$+18$	-18	$+54$	
2	-6	$+6$	-18	$+49$	

Quotient $= 2x^3 - 6x^2 + 6x - 18$; remainder $= 49$:

$$\frac{2x^4 - 12x^2 - 5}{x + 3} = 2x^3 - 6x^2 + 6x - 18 + \frac{49}{x + 3}.$$

In Example 1, on account of the remainder theorem, it follows that $+ 49$ is the value of $(2x^4 - 12x^2 - 5)$ when $x = -3$. This illustrates the following important use of synthetic division:

Rule 2. — To find the value of a polynomial $f(x)$ when $x = r$, divide $f(x)$ by $(x - r)$, by synthetic division; the remainder is $f(r)$.

EXERCISE 86

Divide by long division and also by synthetic division:

1. $(x^3 - 3x + 5) \div (x - 2)$. **2.** $(x^4 + 3x^3 - 5) \div (x + 2)$.

By synthetic division, find the quotient and the remainder:

3. If $2x^3 + 4x^2 + 2x + 1$ is divided by $x - 2$.

4. If $5x^3 - 3x^2 - 2x - 3$ is divided by $x - 3$.

5. If $x^4 - 3x^2 + 2x - 1$ is divided by $x + 3$.

6. If $2x^4 - 3x^3 - 2x + 4$ is divided by $x + 2$.

Solve by use of synthetic division:

7. If $f(x) = x^3 - 3x^2 + 2x - 1$, find $f(2)$; $f(-2)$.

8. If $f(x) = 2x^3 - 3x^2 - 5x + 8$, find $f(4)$; $f(-5)$.

9. If $f(x) = 5x^4 - 3x^3 - 2x + 12$, find $f(3)$; $f(-1)$; $f(-2)$.

10. If $G(z) = 4z^3 - 5z + 9$, find $G(-2)$; $G(\frac{1}{2})$; $G(\frac{1}{4})$; $G(-2)$.

11. Find $(x^4 - 4x^3 + 13x^2 - 36x + 36) \div (x - 2)^2$.

HINT. — Divide twice by $(x - 2)$ by synthetic division.

12. Find $(x^4 + 6x^3 + 6x^2 - 18x - 27) \div (x + 3)^2$.

Prove each statement by use of the factor theorem, and find the other factor by synthetic division:

13. $(x - 1)$ is a factor of $x^7 - 1$.

14. $(x - 2)$ is a factor of $x^5 - 32$.

15. $(x + 2)$ is a factor of $x^5 + 32$.

16. $(x + a)$ is a factor of $x^5 + a^5$.

17. $(x - c)$ is a factor of $(x^n - c^n)$ if n is a positive integer.

145. Graphs of polynomials. — In graphing a polynomial $f(x)$, synthetic division should be used in computing values of $f(x)$.

Example 1. — Graph the function $f(x) = x^3 - 12x + 3$.

Solution. — The following values of $f(x)$ were computed and, in Figure 18, the graph was drawn through the corresponding points:

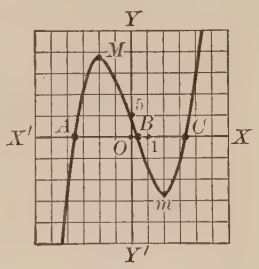


FIG. 18.

WHEN $x =$	-4	-3	-2	-1	0	1	2	3	4
THEN $f(x) =$	-13	$+12$	$+19$	$+14$	$+3$	-8	-13	-6	$+19$

Recall that the real roots of $f(x) = 0$ are the abscissas of the points at which the graph of $f(x)$ meets the x-axis.

Example 2. — Solve graphically: $x^3 - 12x + 3 = 0$.

Solution. — The roots are the abscissas of points A, B, and C in Figure 18. The roots are, approximately, $x = -3.6$, $x = .25$, and $x = 3.3$.

A function $f(x)$ is said to have a **maximum** value for $x = a$ if $f(a)$ is greater than any other value of $f(x)$, for x close to a. A function $f(x)$ has a **minimum** value for $x = a$ if $f(a)$ is less than any other value of $f(x)$, for x close to a.

Thus, from Figure 18, we see that the function $f(x) = (x^3 - 12x - 3)$ has a maximum value for $x = -2$, and a minimum value for $x = 2$,

because the point M is higher than any neighboring point of the graph and the point m is lower than any neighboring point of the graph. M is called a **maximum point** of the graph and m a **minimum point**.

In advanced mathematics it is proved that *the graph of a polynomial $f(x)$ of degree n is a continuous curve, with at most $(n - 1)$ maxima and minima, and with no sharp corners.*

NOTE. — A polynomial $f(x)$ is called a **continuous, single-valued function** of x, because the graph of $f(x)$ is a *continuous* curve, and because, to each value of x, there corresponds a *single* value of $f(x)$.

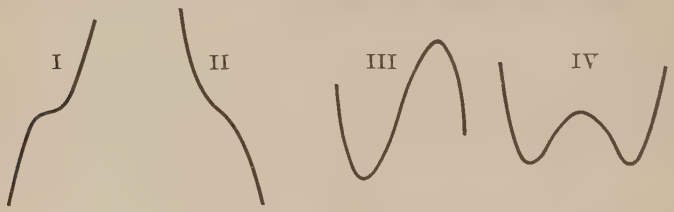


FIG. 19.

NOTE. — Figure 18 and curves I, II, and III in Figure 19 illustrate the different types met as the graphs of cubic functions. Curve IV in Figure 19 is the graph of a certain quartic function. Notice that Curve IV has two minima and one maximum.

EXERCISE 87

Graph each polynomial:

1. $x^3 - x^2 - 9x + 9$.
2. $-x^3 - x^2 + 2x$.
3. $x^3 + 6x + 3x^2 + 8$.
4. x^3.
5. $x^3 + x^2 + 4x - 10$.
6. $x^4 - 14x^2 + 24x + 4$.

Obtain the real roots, graphically:

7. $x^2 - 5x - 3 = 0$.
8. $5x + 8 = 0$.
9. $x^3 + x^2 - 7 - 6x = 0$.
10. $x^3 - 4x^2 - 2x + 8 = 0$.
11. $x^3 - 3x + 1 = 0$.
12. $x^3 - 3x + 4 = 0$.
13. $x^3 - 3x^2 = 2 - 7x$.
14. $x^3 + 6x^2 = 2 - 12x$.
15. $x^4 - x^3 + 15 = 13x^2 - x$.
16. $2x^4 - 7x^2 + 2 = 0$.

146. Fundamental Theorem of Algebra. — *Every integral rational equation, $f(x) = 0$, has at least one root.*

For a proof, see Dickson's *First Course in the Theory of Equations*, p. 155. The fundamental theorem was first proved by a German mathematician, Gauss, in 1799.

Theorem. — Every polynomial $f(x)$ of degree n can be decomposed into a product of n linear factors.

Omission of the proof will not disturb the continuity of the chapter.

Proof. — **1.** Suppose $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, where $a_0 \neq 0$.

2. By the fundamental theorem, the equation $f(x) = 0$ has at least one root. Let r_1 be this root; then, by the factor theorem, $f(x)$ has $(x - r_1)$ as a factor. If $Q_1(x)$ is the other factor, then

$$f(x) = (x - r_1)Q_1(x). \tag{1}$$

3. Since $Q_1(x) = f(x) \div (x - r_1)$, hence $Q_1(x)$ is a polynomial of degree $(n - 1)$ whose term of highest degree is a_0x^{n-1}.

4. By the fundamental theorem, $Q_1(x) = 0$ has a root, r_2. Therefore, $(x - r_2)$ is a factor of $Q_1(x)$, and

$$Q_1(x) = (x - r_2)Q_2(x), \tag{2}$$

where $Q_2(x)$ is a polynomial of degree $(n - 2)$ whose term of highest degree is a_0x^{n-2}. From equations 1 and 2, $f(x) = (x - r_1)(x - r_2)Q_2(x)$.

5. Continuing this process through n steps, we obtain n numbers $r_1, r_2, \cdots, r_n$ and a polynomial $Q_n(x)$ such that

$$f(x) = (x - r_1)(x - r_2) \cdots (x - r_n)Q_n(x), \tag{3}$$

where a_0 is the coefficient of the term of highest degree in $Q_n(x)$. Moreover, the degree of $Q_n(x)$ is $(n - n)$ or zero. That is, $Q_n(x)$ is a constant and hence $Q_n(x) = a_0$. Therefore,

$$f(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_n). \tag{4}$$

Note. — For a given function $f(x)$, all of the numbers $r_1, r_2, \cdots, r_n$ may not be distinct, and some or all may be imaginary.

147. Theorem. — Every equation $f(x) = 0$ of degree n has at most n distinct roots.

Proof. — **1.** From Section 146,

$$f(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_n).$$

Hence, each of $r_1, r_2, \cdots, r_n$ is a root of $f(x) = 0$.

2. If r is any number different from all of $r_1, r_2, \cdots, r_n$, then

$$f(r) = a_0(r - r_1)(r - r_2) \cdots (r - r_n) \neq 0,$$

because no factor of $f(r)$ is zero. Therefore r is not a root of $f(x) = 0$, and hence $r_1, r_2, \cdots, r_n$ are the only roots of $f(x) = 0$.

If a root R occurs only once in the list $r_1, r_2, \cdots, r_n$, then R is called a **simple** root. If R occurs exactly h times among $r_1, r_2, \cdots, r_n$, or in other words, *if* $(x - R)^h$ *is the highest power of* $(x - R)$ *which is a factor of* $f(x)$, then R is called **a multiple** root of $f(x) = 0$, whose **multiplicity** is h.

The preceding theorem may be restated as follows:

Every equation of degree n has exactly n roots $r_1, r_2, \cdots, r_n$, where a root of multiplicity h is counted as h roots.

NOTE. — Roots of multiplicities 2 and 3 are called **double** and **triple** roots, respectively.

Corollary 1: supplementary. — *If two polynomials*

$$a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

and

$$b_0x^n + b_1x^{n-1} + \cdots + b_n,$$

each of degree not greater than n, are equal in value for more than n distinct values of x, then the polynomials are identical, term by term; that is, $a_0 = b_0$, $a_1 = b_1$, $\cdots$, $a_n = b_n$, *and hence the polynomials are equal for all values of x.*

Proof. — **1.** By assumption, the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_n - (b_0x^n + b_1x^{n-1} + \cdots + b_n) = 0, \quad (1)$$

or,

$$(a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + \cdots + (a_n - b_n) = 0, \quad (2)$$

has more than n distinct roots.

2. If any one of the coefficients $(a_0 - b_0)$, $(a_1 - b_1)$, $\cdots$, $(a_n - b_n)$ in (2) were not zero, then (2) would be an equation of degree n, or less,

with more than n distinct roots. This fact would contradict the preceding theorem. Hence, all coefficients in (2) must be zero; that is, $a_0 = b_0$, $a_1 = b_1$, $\cdots$, $a_n = b_n$.

148. Theorem. — If an imaginary number $(a + bi)$ is a root of an equation $f(x) = 0$, with real coefficients, then the conjugate imaginary, $(a - bi)$, is also a root of $f(x) = 0$. That is, **imaginary roots occur in pairs.**

Omission of the proof will not disturb the continuity of the chapter.

Proof. — **1.** Let $D(x) = [x - (a + bi)][x - (a - bi)]$

$$= x^2 - 2ax + a^2 + b^2.$$

2. Divide $f(x)$ by $D(x)$ until the remainder is a linear function, $cx + d$, and let $Q(x)$ be the quotient. Then,

$$f(x) \equiv D(x)\,Q(x) + cx + d, \tag{1}$$

where c and d are real numbers because $D(x)$ is a polynomial with real coefficients.

3. Since $a + bi$ is a root of $f(x) = 0$, hence $f(a + bi) = 0$. Since $D(x)$ has the factor $[x - (a + bi)]$, hence $D(a + bi) = 0$. Therefore, when we place $x = a + bi$ in (1), we obtain $0 = 0 + c(a + bi) + d$, or

$$(ac + d) + (bc)i = 0. \tag{2}$$

4. By use of statement 1 in Section 131, it follows from (2) that

$$ac + d = 0 \text{ and } bc = 0.$$

5. By hypothesis, $(a + bi)$ is imaginary and hence $b \neq 0$. Since $bc = 0$, hence $c = 0$.

6. Since $c = 0$, and $ac + d = 0$, therefore $d = 0$.

7. Therefore, the remainder $cx + d$ in (1) is identically zero, and thus $D(x)$ is a factor of $f(x)$. Hence, $f(x)$ has the factor $[x - (a - bi)]$ because this is a factor of $D(x)$. Hence, $f(x)$ is zero when $x = a - bi$, or $(a - bi)$ is a root of $f(x) = 0$.

Corollary 1. — *Every polynomial $f(x)$ with real coefficients can be expressed as a product of real linear and quadratic factors.*

Proof. — **1.** If $r_1, r_2, \cdots, r_n$ are the roots of $f(x) = 0$, then

$$f(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_n). \tag{3}$$

2. By the preceding theorem, if $f(x)$ has an imaginary factor $[x-(a+bi)]$, then $[x-(a-bi)]$ is also a factor. Since

$$[x-(a+bi)][x-(a-bi)] = x^2 - 2ax + a^2 + b^2,$$

which is a quadratic expression with real coefficients, hence all imaginary factors can be combined by pairs to give real quadratic factors.

NOTE. — Corollary 1 is true, theoretically, but, practically, we can factor $f(x)$ only when it is possible to completely solve $f(x) = 0$.

In the future, unless otherwise stated, polynomials considered will be supposed to have real coefficients.

Example 1. — Form an equation with the following roots and no others: -2, 4 as a triple root, $(3 + i\sqrt{2})$ and $(3 - i\sqrt{2})$.

Solution. — 1. One equation is

$$(x+2)(x-4)^3[x-(3+i\sqrt{2})][x-(3-i\sqrt{2})] = 0,$$

or,
$$(x+2)(x-4)^3(x^2-6x+11) = 0. \qquad (4)$$

If the left member of (4) is multiplied by any constant, except zero, the new equation also has the given roots.

Example 2. — Solve: $x(x+5)(x^2+4x+7) = 0$.

Solution. — 1. If $x^2 + 4x + 7 = 0$,

$$x = \tfrac{1}{2}(-4 \pm \sqrt{-12}) = -2 \pm i\sqrt{3}.$$

2. If $x + 5 = 0$, then $x = -5$. $x = 0$ is the fourth root.

EXERCISE 88

Solve:

1. $(x-2)(x+5)(x+7) = 0$. 2. $(x+4)(x-3)(x+8) = 0$.
3. $(x^2 - 3x)(x^2 - 5x + 8) = 0$.
4. $(x^2 + 4x)(3x^2 + 7x + 9) = 0$.
5. $(x^2+9)(9x^2+16) = 0$. 6. $(4x^2+9)(2x-3) = 0$.
7. $(9x^2 + 12x + 4)(x^2 + 4) = 0$.
8. $(4x^2 + 20x + 25)(x^2 - 8) = 0$.

Form an equation with the given numbers, and no others, as roots:

9. $2, 2, -5, 4$.

10. $4, 3, -2, -2$.

11. $(1-\sqrt{2}), (1+\sqrt{2}), 3$.

12. $(2-\sqrt{3}), (2+\sqrt{3}), 4$.

13. $3i, -3i, \frac{5}{4}, 6$.

14. $4i, -4i, \frac{2}{3}, \frac{2}{3}$.

15. $(2+2i), (2-2i), 3$.

16. $(3+4i), (3-4i), 4, 5$.

17. $(2+i\sqrt{3}), (2-i\sqrt{3}), 2$.

18. $(3-i\sqrt{6}), (3+i\sqrt{6}), 5$.

19. $\sqrt{2}, -\sqrt{2}, (4+i\sqrt{5}), (4-i\sqrt{5})$.

20. $(i\sqrt{3}+2), (-i\sqrt{3}+2), 5, -5$.

21. Form a quartic equation with 2 as a double root, and with 3 and -4 as simple roots.

If $f(x)=0$ is an equation with real coefficients, and if $f(x)=0$ has the following number as a root, what other number is a root of $f(x)=0$?

22. $3+5i$. **23.** $-2+6i$. **24.** $-3-7i$. **25.** $8-3i$.

26. Prove that a cubic equation, with real coefficients, has either three real roots, or else one real and two imaginary roots.

27. Prove that a quartic equation $f(x)=0$ with real coefficients has either four real roots, or else two real and two imaginary roots, or else four imaginary roots.

SUPPLEMENTARY PROBLEMS

28. State and prove theorems similar to those of problems 26 and 27 for equations of degree 5, and of degree 6.

29. Prove that every equation of odd degree with real coefficients has at least one real root.

30. Show that 2 is a double root of

$$x^4 - 4x^3 + 13x^2 - 36x + 36 = 0,$$

and find the other roots.

31. Show that -3 is a double root of

$$x^4 + 6x^3 + 6x^2 - 18x - 27 = 0,$$

and find the other roots.

32. Prove that, if an equation with rational coefficients has a root $a+\sqrt{b}$ where a and b are rational but $\sqrt{b}$ is irrational,

then the equation also has the root $a - \sqrt{b}$. Use the method of the proof on page 211.

149. Graph of a factored polynomial; supplementary. If r is real, and if $(x - r)^h$ is the highest power of $(x - r)$ which is a factor of $f(x)$, then the following statements are true:

I. If h is an odd integer, the graph of $f(x)$ crosses the x-axis at $x = r$. If $h = 1$, the graph cuts the x-axis sharply at $x = r$ (see I_a in Figure 20). If h is an odd integer greater than 1, the graph on each side of $x = r$ is tangent to the x-axis at $x = r$ (see I_b in Figure 20).

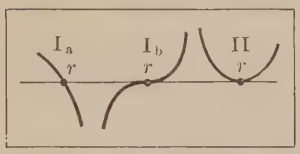


FIG. 20.

II. If h is an even integer, the graph of $f(x)$ is entirely on one side of the x-axis, near $x = r$, and is tangent to the x-axis at $x = r$ (see II in Figure 20).

NOTE. — For any given polynomial $f(x)$, the preceding statements concerning the shape of the graph at the points where it meets the x-axis could be verified by accurately plotting a great many points of the graph. But if statements I and II are admitted, then by use of them we can easily obtain a good approximation to the graph of a factored polynomial $f(x)$ if all of its factors are real and linear.

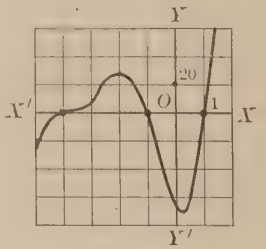


FIG. 21.

Example 1. — Graph the function

$$f(x) = (x + 4)^3 (x + 1)(x - 1).$$

Solution. — 1. The roots of $f(x) = 0$ are -4, -1, and 1. Hence, the graph meets the x-axis at $x = -4$, $x = -1$, and $x = 1$. The graph of $f(x)$ in Figure 21 was drawn by use of the following values of $f(x)$.

WHEN $x =$	-5	-4	-3	-2	-1	0	1	2
THEN $f(x) =$	-24	0	8	24	0	-64	0	648

NOTE. — If $f(x)$ has a quadratic factor which is the simplified form of a product $[x - (a + bi)][x - (a - bi)]$ corresponding to a pair of

imaginary roots of $f(x) = 0$, this quadratic factor is zero only when $x = a + bi$ and $x = a - bi$. Hence, corresponding to this quadratic factor there are no intersections of the graph of $f(x)$ with the x-axis because the factor is zero for no real values of x. The graph of

$$f(x) = (x + 3)(x^2 - 2x + 5)$$

in Figure 22 illustrates these remarks. The graph meets the axis only at $x = -3$, because the roots of the equation $x^2 - 2x + 5 = 0$ are $x = 1 \pm 2i$.

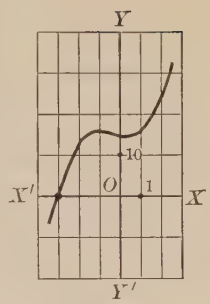


FIG. 22.

EXERCISE 89

Graph each function, computing values of the functions without expanding the products:

1. $(x + 5)$. **2.** $(x - 4)^2$. **3.** $(x + 2)^3$. **4.** $(x - 5)^4$.

5. $(x + 5)(x + 4)$.

6. $x(x - 3)(x - 5)$.

7. $(x - 3)^2(x - 4)$.

8. $(3x + 2)^2(x - 3)$.

9. $(x + 2)(x^2 + 3x + 2)$.

10. $(x + 3)(x^2 + 2x - 3)$.

11. $x(x + 2)^3$.

12. $(2x - 5)^4(x - 6)^3$.

13. $(x^2 - 3x - 4)(x - 1)^2$.

14. $(2x^2 + 3x - 2)(x + 2)^2$.

150. Transformation of an equation sometimes is of aid in obtaining information about the roots. Let $r_1, r_2, \cdots, r_n$ be the roots of $f(x) = 0$. Then, if we substitute $x = -X$ in $f(x) = 0$, we obtain an equation in X, $f(-X) = 0$, which is satisfied when and only when $-X = r_1$, or $-X = r_2, \cdots$, or $-X = r_n$. Hence, the roots of $f(-X) = 0$ are $X = -r_1$, $X = -r_2, \cdots, X = -r_n$.

Rule 1. — *To obtain the equation $f(-X) = 0$, whose roots are the negatives of the roots of a given equation $f(x) = 0$, change the sign of the coefficient of each term of odd degree in $f(x) = 0$, and then change x to X.*

Proof. — If we replace x by $-X$ in the given equation

$$f(x) = a_n + a_{n-1}x + a_{n-2}x^2 + a_{n-3}x^3 + \cdots + a_0x^n = 0,$$

we obtain $f(-X) = 0$, or

$a_n + a_{n-1}(-X) + a_{n-2}(-X)^2 + a_{n-3}(-X)^3 + \cdots + a_0(-X)^n = 0,$

or, $\quad a_n - a_{n-1}X + a_{n-2}X^2 - a_{n-3}X^3 + \cdots \text{ etc.} = 0, \qquad (1)$

which has the form specified in Rule 1.

Example 1. — Obtain an equation whose roots are the negatives of the roots of $x^4 + 3x^3 - 3x^2 - 5x + 8 = 0$.

Solution. — By Rule 1, the desired equation is

$$X^4 - 3X^3 - 3X^2 + 5X + 8 = 0.$$

EXERCISE 90

Find an equation whose roots are the negatives of the roots of the given equation:

1. $2x^3 - 3x^2 - 5x + 2 = 0$. **4.** $x^4 - x^3 + x^2 - 5 = 0$.

2. $3x^3 - 2x^2 + 3x - 4 = 0$. **5.** $x^8 - x^7 - 3 = 0$.

3. $2x^4 - 5x^2 + 2x - 1 = 0$. **6.** $-2x^4 - 5x^3 + 125 = 0$.

7. How are the roots of $f(x) = 0$ related to the roots of (*a*) $f(2X) = 0$; (*b*) $f(X + 3) = 0$; (*c*) $f(-2X) = 0$?

151. Descartes' rule of signs; supplementary. — In a polynomial $f(x)$, arranged in descending powers of x, if two successive terms differ in sign, there is said to be a **variation** in sign. In counting variations, zero terms are disregarded.

Thus, there are three variations of sign in $x^8 - x^2 + 2x - 3$.

Auxiliary Theorem. — *If $g(x)$ is any polynomial and if $r > 0$, then $(x - r)g(x)$ has at least one more variation of sign than $g(x)$.*

Proof. — **1.** Suppose that*

$$g(x) = + + + + - - - - + + + + - - -, \qquad (1)$$

where in place of each term we have written merely its sign.

* The statements of this proof are general, in their application, even though they refer to a special polynomial whose signs are given in (1). In reading the proof for the first time, assume that in $g(x)$ no power of x is missing.

Then, $xg(x) = + + + + - - - - + + + + - - -$, (2)
and, $-rg(x) = \quad - - - - + + + + - - - - + + +$. (3)
$(x - r)g(x) = + \pm \pm \pm - \pm \pm \pm + \pm \pm \pm - \pm \pm +$, (4)

To obtain (4), we added corresponding members of (2) and (3).

2. The first of each group of " + " terms in (2) is part of a " + " term in (4). The first of each group of " − " terms in (2) is part of a " − " term in (4). The intermediate terms are indicated " ± " in (4) because each one is +, or −, or zero, depending on the values of r and of certain coefficients in $g(x)$.

3. We shall underestimate the number of variations of sign in $(x - r)g(x)$ if we assume, in (4), that each ambiguous sign is the same as the preceding unambiguous sign. Under this assumption, the signs of $(x - r)g(x)$ are

$$+ + + + - - - - + + + + - - - +, \qquad (5)$$

which are the same as the signs in $g(x)$, with an additional sign at the right end. This additional sign is unlike the last sign in $g(x)$, because the last term in $(x - r)g(x)$ equals the last term in $g(x)$ multiplied by $-r$.

4. Hence, in (5) we have all variations of sign of $g(x)$, with one additional variation due to the last sign in (5). Therefore, $(x - r)g(x)$ has at least one more variation of sign than $g(x)$.

Illustration. In $g(x) = x^4 - 2x^3 + x^2 + 2x - 5$, there are 3 variations of sign; $(x - 2)g(x) = x^5 - 4x^4 + 5x^3 - 9x + 10$, which has 4 variations of sign.

NOTE. — If certain powers of x are missing in $g(x)$, the preceding proof holds without alteration, if a zero is listed in (1), (2), and (3) for each missing power.

Descartes' rule of signs. — The number of positive roots of $f(x) = 0$ cannot exceed the number of variations of sign in $f(x)$.

Proof. — **1.** Let $r_1, r_2, \cdots, r_k$ be the positive roots of $f(x) = 0$. Then, by the factor theorem,

$$f(x) = (x - r_1)(x - r_2) \cdots (x - r_k)Q(x),$$

where $\quad Q(x) = f(x) \div [(x - r_1)(x - r_2) \cdots (x - r_k)]$.

2. By the auxiliary theorem, $(x - r_1)Q(x)$ has at least one more variation of sign than $Q(x)$, or at least 1 variation of sign. Similarly, $(x - r_2)(x - r_1)Q(x)$ has at least one more variation of sign than $(x - r_1)Q(x)$, or, at least 2 variations. Finally, $(x - r_1)(x - r_2) \cdots (x - r_k)Q(x)$, or $f(x)$ itself, has at least k variations of sign, where k is the number of positive roots of $f(x) = 0$.

The roots of $f(-x) = 0$ are the negatives of the roots of $f(x) = 0$. Hence, the *negative* roots of $f(x) = 0$ are the *positive* roots of $f(-x) = 0$. Therefore, from Descartes' rule, we obtain the following corollary.

Corollary 1. — *The number of negative roots of* $f(x) = 0$ *cannot exceed the number of variations of sign in* $f(-x)$.

EXERCISE 91

State what can be learned about the roots by use of Descartes' rule of signs and, if possible, tell whether any roots are imaginary.

Example. — Given equation: $f(x) = 2x^4 + 5x^2 - 4x - 1 = 0$.

Solution. — 1. $f(x)$ has one variation of sign. Hence there is at most one positive root.

2. $f(-x) = 2x^4 + 5x^2 + 4x - 1$, which has one variation of sign. Hence, $f(x) = 0$ has at most one negative root.

3. Since $f(x) = 0$ has four roots, it follows from Steps 1 and 2 that at least two roots must be imaginary.

1. $2x^3 + 3x^2 - 7x + 4 = 0$.

2. $4x^3 - 5x^2 - 8x - 5 = 0$.

3. $x^3 + 3x - 4 = 0$.

4. $2x^3 + 5x + 8 = 0$.

5. $x^4 + 3x^2 - 5x - 1 = 0$.

6. $x^4 - 3x^3 + 2x^2 + 1 = 0$.

7. $x^5 + 3 = 0$.

8. $x^5 - 1 = 0$.

9. $x^4 + 3x^2 + 5 = 0$.

10. $x^6 + 4x^4 + 3 = 0$.

11. $x^4 - 3x - 3 = 0$.

12. $2x^6 - 4x^2 - 5x + 6 = 0$.

Given that the roots are all real, determine their signs:

13. $x^3 - 4x^2 - 2x = -8$.

14. $x^3 + x^2 - 6x = 6$.

Prove that each equation has exactly two imaginary roots.

15. $x^4 + x^2 + 4x - 5 = 0$. **16.** $2x^4 - 3x^3 - 4x - 2 = 0$.

HINT. — Show by a rough graph that there is a positive root.

17. $3x^3 + 4x - 5 = 0$. **18.** $5x^3 + 3x^2 + 5 = 0$.

152. Limits for real roots. — If all real roots of $f(x) = 0$ are greater than l and less than L, we call l a **lower limit** and L an **upper limit** for the real roots of $f(x) = 0$. An upper limit can be found by use of the following rule, which is easily justified in each application; a lower limit for the roots of $f(x) = 0$ can be found by applying the rule to the equation $f(-x) = 0$, whose roots are the negatives of the roots of $f(x) = 0$.

Rule 1. — *If k is positive, and if all numbers are positive in the third line of the synthetic division of $f(x)$ by $(x - k)$, then k is greater than any real root of $f(x) = 0$.*

NOTE. — Before applying Rule 1 to an equation $f(x) = 0$, where the coefficient of the highest power of x is negative, multiply both members by -1, to make the coefficient of the highest power positive.

Example 1. — Find limits for the roots of

$$f(x) = x^3 + 3x^2 - 12x - 9 = 0.$$

Solution. — 1. On dividing $f(x)$ by $(x - 3)$, all numbers in the third line are found to be positive, and $f(3) = 9$.

	1	3	−12	−9	| 3
		3	18	18	
	1	6	6	9	

Hence, if we should divide $f(x)$ by $(x - a)$, where a is any number greater than 3, we would find that $f(a) > 9$, because each number in the third row in this new division would be greater than the corresponding number in the division by $(x - 3)$. Hence, if $a > 3$, then a is not a root of $f(x) = 0$; or, 3 is an upper limit for the real roots of $f(x) = 0$.

2. To find a lower limit, consider $f(-x) = -x^3 + 3x^2 + 12x - 9 = 0$, or, $x^3 - 3x^2 - 12x + 9 = 0$. Synthetic division of $(x^3 - 3x^2 - 12x + 9)$ by $(x - 6)$ shows that 6 is an upper limit for the roots of $f(-x) = 0$. Hence, -6 is *less than* any root of $f(x) = 0$, or, -6 is a lower limit for the roots.

	1	−3	−12	9	| 6
		6	18	36	
	1	3	6	45	

EXERCISE 92

Find limits for the real roots of each equation:

1. $x^3 + 2x^2 - 10x + 3 = 0$.
2. $2x^3 + 3x^2 - 15x + 8 = 0$.
3. $5x^4 - 3x^3 - 20x^2 - 50 = 0$.
4. $x^5 - 2x^4 - 32 = 0$.
5. $x^4 - 220x + 8 = 0$.
6. $2x^5 - 4x^3 - 3x^2 - 40 = 0$.
7. $4x^4 - 20x^2 - 3x - 2 = 0$.
8. $x^3 - 2x^2 - 40x + 15 = 0$.
9. Prove Rule 1 for any equation $f(x) = 0$.

HINT. — Dividing $f(x)$ by $(x - k)$ gives a quotient $Q(x)$ and a remainder R, such that $f(x) = (x - k)Q(x) + R$. By the hypothesis of Rule 1, R and all coefficients of $Q(x)$ are positive. Use these facts to show that, if $a > k$, then $f(a) > 0$, and hence a is not a root of $f(x) = 0$.

153. Theorem on rational roots. — *If an equation*

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0, \quad (1)$$

with integral coefficients, has a rational root $\frac{c}{d}$, *where* $\frac{c}{d}$ *is a fraction in its lowest terms, then* c *is a factor of* a_n *and* d *is a factor of* a_0.

Proof. — **1.** By hypothesis, c and d are integers with no common factor except ± 1. On substituting $x = \frac{c}{d}$ in (1) we obtain

$$a_0\frac{c^n}{d^n} + a_1\frac{c^{n-1}}{d^{n-1}} + a_2\frac{c^{n-2}}{d^{n-2}} + \cdots + a_{n-1}\frac{c}{d} + a_n = 0. \quad (2)$$

2. On multiplying both sides of (2) by d^n we obtain

$$a_0c^n + a_1c^{n-1}d + a_2c^{n-2}d^2 + \cdots + a_{n-1}cd^{n-1} + a_nd^n = 0, \quad (3)$$

or, $$d(a_1c^{n-1} + a_2c^{n-2}d + \cdots + a_{n-1}cd^{n-2} + a_nd^{n-1}) = -a_0c^n. \quad (4)$$

3. In (4), all letters represent integers, and d is a factor of the left side. Hence, d must be a factor of the right side, $-a_0c^n$.

But, unless $d = \pm 1$, d is not a factor of c^n because d is not a factor of c. Hence, d is a factor of a_0.

4. From (3) we obtain

$$c(a_0c^{n-1} + a_1c^{n-2}d + a_2c^{n-3}d^2 + \cdots + a_{n-1}d^{n-1}) = -a_nd^n. \quad (5)$$

5. In (5), c is a factor of the left side. Hence, c is a factor of $-a_nd^n$. But, unless $c = \pm 1$, c is not a factor of d^n. Hence, c is a factor of a_n.

Corollary 1. — *Any rational root of an equation*

$$x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_{n-1}x + b_n = 0, \quad (6)$$

with integral coefficients, is an integer and an exact divisor of b_n.

Proof. — By the previous theorem, if $x = \frac{c}{d}$ is a root of (6), then d is a factor of the coefficient of x^n, and c is a factor of b_n. Since the coefficient of x^n is 1, hence $d = \pm 1$, and therefore the rational root $\frac{c}{d}$ is an integer, $\pm c$, which is a factor of b_n.

If an equation is in the form of (6) we shall say that it is in the ***b*-form.** The essential feature of (6) is that the coefficient of x^n is 1.

NOTE. — In solving an equation, whenever a rational root is found, depress the degree of the original equation $f(x) = 0$ by removing the factor of $f(x)$ corresponding to the known root. Then, continue the solution by finding the roots of the **depressed equation.**

Example 1. — Find all rational roots of

$$f(x) = x^4 - 6x^3 + 3x^2 + 24x - 28 = 0.$$

Solution. — 1. By Corollary 1, the possible rational roots are the integral divisors of -28, or ± 1, ± 2, ± 4, ± 7, ± 14, and ± 28.

2. By synthetic division of $f(x)$ by $(x - 1)$, we find $f(1) = -6$; hence, 1 is not a root. Similarly we find that $f(-1) = -42$, and hence -1 is not a root.

3. By synthetic division by $(x - 2)$, we find $f(2) = 0$, and $f(x) = (x-2)(x^3-4x^2-5x+14)$.

	1	−6	3	24	−28	2
		2	−8	−10	28	
	1	−4	−5	14	0	

Hence, $x = 2$ is a root. The other roots of $f(x) = 0$ are the roots of the depressed equation $x^3 - 4x^2 - 5x + 14 = 0$.

4. Let $Q(x) = x^3 - 4x^2 - 5x + 14$. The possible rational roots of $Q(x) = 0$ are ± 1, ± 2, ± 7, ± 14. From Step 2, we know that ± 1 are not roots. By synthetic division of $Q(x)$ by $(x + 2)$, we find that $Q(-2) = 0$, and that $Q(x) = (x + 2)(x^2 - 6x + 7)$. Hence, $x = -2$ is a root. The depressed equation is $x^2 - 6x + 7 = 0$, whose solutions are $x = 3 \pm \sqrt{2}$, which are irrational. Hence, $x = 2$ and $x = -2$ are the only rational roots of $f(x) = 0$. Notice that, incidentally, we have found *all* of the roots of $f(x) = 0$.

	1	−4	−5	14	−2
		−2	12	−14	
	1	−6	7	0	

Example 2. — Find all rational roots of

$$f(x) = 3x^3 + 2x^2 - 3x - 2 = 0.$$

Solution. — 1. By the theorem on rational roots, if $x = \frac{c}{d}$ is a root, then the possible values of c are ± 1 and ± 2; the possible values of d are ± 1 and ± 3. Forming all possible quotients $\frac{c}{d}$ from these values, we find the following as the possible rational roots: ± 1, ± 2, $\pm \frac{1}{3}$, $\pm \frac{2}{3}$.

2. By synthetic division by $(x - 1)$ we find that $f(1) = 0$, and that $f(x) = (x - 1)(3x^2 + 5x + 2)$. Hence, $x = 1$ is a root, and the depressed equation is $3x^2 + 5x + 2 = 0$. Solving this by the quadratic formula, we find that the other roots are $x = -1$ and $x = -\frac{2}{3}$.

	3	2	−3	−2	1
		3	5	2	
	3	5	2	0	

EXERCISE 93

Find all rational roots and, if the determination of these leads to a depressed equation which is a quadratic, find all of the roots:

1. $x^3 + 3x^2 - 33x - 35 = 0$.
2. $x^3 + 5x^2 - 8x - 12 = 0$.
3. $x^3 + 8x^2 + 22x + 21 = 0$.
4. $x^3 + 5x^2 + 11x + 10 = 0$.
5. $3x^3 + 7x^2 - 3x - 2 = 0$.
6. $5x^3 + 17x^2 - 4x - 4 = 0$.
7. $x^4 - 4x^3 + 6x^2 - 8x + 8 = 0$.
8. $x^4 - 45x^2 + 40x + 84 = 0$.
9. $2x^3 - 9x^2 - 2x + 30 = 0$.

10. $4x^4 + 8x^3 - 13x^2 - 18x + 9 = 0.$

11. $16x^3 + 2x^2 + 8x + 1 = 0.$

12. $x^5 - 4x^4 - 5x^3 + 20x^2 + 4x = 16.$

13. $z^4 - 3z^3 - z^2 + 15z = 20.$

14. $4x^3 + 16x^2 - 19x + 5 = 0.$

15. $y^4 + 2y^3 - 4y^2 + 10y = 5.$

SUPPLEMENTARY PROBLEMS

Find limits for the roots and then obtain all rational roots, making use of the limits in rejecting possible values:

16. $x^4 + x^3 - 25x^2 - 27x = 54.$

17. $x^3 + x^2 - 18x + 84 = 0.$

18. $x^3 + 8x^2 - 28x - 215 = 0.$

19. $x^4 + 3x^3 - 19x^2 - 26x + 93 = 0.$

20. $x^3 - 3x^2 + 4x - 240 = 0.$

21. $x^3 - 2x^2 - 32x + 88 = 0.$

22. $2x^4 - 4x^2 - 3x + 60 = 0.$

154. Transformation to multiply the roots; supplementary. — If we substitute $x = \frac{y}{m}$ in $f(x) = 0$, we obtain $f\left(\frac{y}{m}\right) = 0$, each of whose roots is m times a root of $f(x) = 0$.

Thus, if $x = 3$ is a root of $f(x) = 0$, then $f\left(\frac{y}{m}\right) = 0$ is satisfied if $\frac{y}{m} = 3$, or $y = 3m$.

Rule 1. — *To obtain the equation* $f\left(\frac{y}{m}\right) = 0$, *each of whose roots is* m *times a root of* $f(x) = 0$, *where*

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n, \qquad (1)$$

multiply the successive coefficients in $f(x) = 0$, *starting with* a_1, *by* m, m^2, m^3, $\cdots$, m^n, *respectively, and replace* x *by* y.

Proof. — **1.** On replacing x by $\frac{y}{m}$ in $f(x) = 0$, we obtain

$$f\left(\frac{y}{m}\right) = a_0\left(\frac{y}{m}\right)^n + a_1\left(\frac{y}{m}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{y}{m}\right) + a_n = 0. \qquad (2)$$

2. On multiplying both sides of (2) by m^n, we obtain

$a_0y^n + a_1my^{n-1} + a_2m^2y^{n-2} + \cdots + a_{n-1}m^{n-1}y + a_nm^n = 0$,

which completes the proof of Rule 1.

NOTE. — In applying Rule 1, supply each missing power in $f(x)$ with zero as a coefficient.

Example 1. — Obtain an equation each of whose roots is three times a root of $2x^3 + 3x^2 - 5 = 0$.

Solution. — By Rule 1 the desired equation is

$$2y^3 + 2 \cdot 3y^2 - 5 \cdot 3^3 = 0, \text{ or } 2y^3 + 6y^2 - 135 = 0.$$

By use of Rule 1, we obtain a new method for finding all rational roots of an equation with integral coefficients of the form $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, where $a_0 \neq 1$.

Example 2. — Find all rational roots of

$$64x^3 - 16x^2 + 12x - 3 = 0.$$

Solution. — 1. Divide both sides by 64: $x^3 - \frac{x^2}{4} + \frac{3x}{16} - \frac{3}{64} = 0.$ (3)

2. *Transform to obtain an equation each of whose roots is* 4 *times a root of* (3). By Rule 1, the transformed equation is

$$y^3 - y^2 + 3y - 3 = 0. \qquad (4)$$

A root x of (3) and the corresponding root y of (4) are related by the equation $x = \frac{1}{4}y$.

3. Equation 4 is in the *b*-form, and its only rational roots are integers. By the method of Example 1, page 221, we find that the only rational root of (4) is $y = 1$. Since $x = \frac{1}{4}y$, the only rational root of (3) is $x = \frac{1}{4}$.

Discussion. — In Step 2, we transformed the original equation to multiply the roots by the *smallest integer* which would cause the resulting equation to be in the *b*-form with *integral* coefficients.

EXERCISE 94

Transform each equation into another whose roots are those of the given equation multiplied by the number in parentheses.

1. $2x^3 + 3x^2 - 5x + 2 = 0.$ (2)
2. $3x^3 - 2x^2 + 3x - 4 = 0.$ (4)
3. $2x^4 - 5x^2 + 2x - 1 = 0.$ (-2)
4. $x^4 - \frac{1}{4}x^3 + \frac{3}{16}x^2 - 1 = 0.$ (4)
5. $x^3 - \frac{1}{3}x^2 + \frac{2}{3}x - 5 = 0.$ (-3)
6. $x^3 - \frac{x^2}{3} - \frac{x}{36} + \frac{1}{108} = 0.$ (6)

Find all rational roots by the method of Example 2, page 224:

7. $2x^3 - 43x^2 - 41x - 90 = 0.$
8. $2x^3 + 7x^2 + 13x + 20 = 0.$
9. $4x^4 + 9x^3 + 12x + 27 = 0.$
10. $9x^4 + 18x^3 - 13x^2 - 8x + 4 = 0.$
11. $108y^3 + 252y^2 + 3y + 7 = 0.$
12. $12x^3 + 16x^2 - 3x - 4 = 0.$

155. Location of real roots of $f(x) = 0$. — The following theorem states a principle by which real roots of $f(x) = 0$ can be located, *without the aid of a graph of* $f(x)$.

Theorem I. — *If a and b are real numbers, $a < b$, for which $f(a)$ and $f(b)$ have unlike signs, then $f(x) = 0$ has at least one root between $x = a$ and $x = b$.*

Illustration. Consider $f(x) = x^3 + 6x + 5 = 0$. We find that $f(-1) = -2$ and $f(1) = 12$. Hence, by Theorem I, $f(x) = 0$ has at least one root between $x = -1$ and $x = 1$.

Proof of Theorem I. — On the graph of $f(x)$ (see Figure 23) the points P and Q, corresponding to $x = a$ and $x = b$, respectively, will be on opposite sides of the x-axis, because $f(a)$ and $f(b)$ have unlike signs. Since the graph is a continuous curve joining P and Q, the graph must cross the x-axis at least once between P

and Q. To each intersection with the x-axis there corresponds a root of $f(x) = 0$. Hence, there is *at least one* root between $x = a$ and $x = b$.

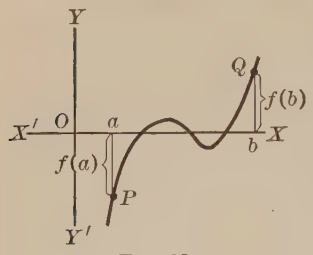


FIG. 23.

156. Irrational roots* obtained by successive graphs.

Example 1. — Solve

$$f(x) = x^3 - 3x^2 - 2x + 5 = 0.$$

Solution. — **1.** We find that there are no rational roots. By the method of Section 152, we obtain -2 as a lower limit and 4 as an upper limit for real roots. Hence, we plot a rough graph of $f(x)$ from $x = -2$ to $x = 4$. The roots are approximately -1.3, 1.2, and 3.1, as read from the graph, in Figure 24.

WHEN $x =$	-2	-1	0	1	2	3	4
THEN $f(x) =$	-11	3	5	1	-3	-1	13

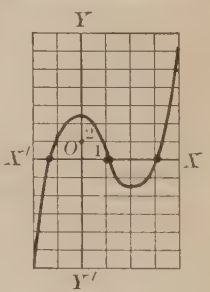


FIG. 24.

2. To obtain accurately the root which is approximately 3.1.

a. Locate the root between successive tenths. — By synthetic division we compute $f(3.1) = -.24$ and $f(3.2) = +.65$. Hence, by Theorem I of Section 155, there is a root between 3.1 and 3.2.

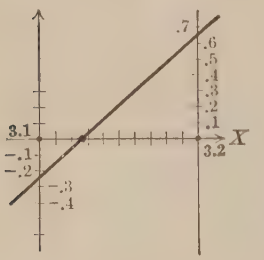


FIG. 25.

WHEN $x =$	3.1	3.2
THEN $f(x) =$	$-.24$	$+.65$

b. Graph $f(x)$ from $x = 3.1$ to $x = 3.2$, by use of $f(3.1)$ and $f(3.2)$. The graph, in Figure 25, is a straight line because we use only two points to determine it. If more points were used, corresponding to values of x between 3.1 and 3.2, the curved line, which is the actual graph of $f(x)$, could be more accurately

* This section may be omitted if Horner's method is to be studied. If Horner's method is to be omitted, proceed to Section 159 after Section 156.

determined. The approximate graph in Fig. 25 is sufficient for our purpose. From this graph, we read that the root is approximately 3.13.

c. Locate the root between successive hundredths. — We find, as in Step *a*, that the

When x =	3.12	3.13
then $f(x)$ =	− .073	+ .014

root is between $x = 3.12$ and $x = 3.13$. From a graph of $f(x)$ between $x = 3.12$ and $x = 3.13$, in Figure 26, we read that the root is approximately 3.128.

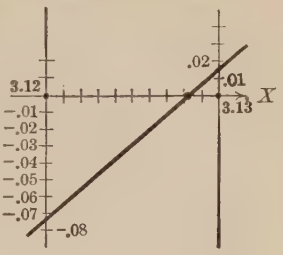


Fig. 26.

3. Similarly, we obtain the other roots: $x = 1.202$ and $x = -1.330$.

Note. — In Example 1, the fourth digit is in doubt in each of the final values of the roots. We could obtain the fourth digit accurately in any root by performing one more step of the process used.

The solution of Example 1 illustrates the following method.

Method for determining an irrational root of $f(x) = 0$ by successive graphs:

1. *From the graph of $f(x)$, estimate the root to the nearest tenth and call this estimated value x_1.*

2. *Locate the root between successive tenths by computing $f(x)$ for $x = x_1$ and for the successive tenths near x_1.*

3. *Graph $f(x)$, with an* ***enlarged scale****, for values of x between the tenths where the root lies, using a straight line for the curve. From the graph, read the apparent value x_2 of the root to the nearest hundredth.*

4. *Locate the root between successive hundredths by computing values of $f(x)$. Continue, by the method of Steps 2 and 3, until the root is obtained as accurately as is desired.*

157. Transformation to decrease the roots of an equation.* — Consider the equation

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0. \qquad (1)$$

* This section is auxiliary to Horner's method for irrational roots.

If we substitute $x = (X + h)$ in $f(x) = 0$, we obtain

$$f(X + h) = a_0(X + h)^n + a_1(X + h)^{n-1} + \cdots + a_{n-1}(X + h) + a_n = 0, \qquad (2)$$

which is an equation in X whose roots are those of $f(x) = 0$ each reduced by the constant h. If each power of $(X + h)$ in (2) is expanded by the binomial theorem, the term in X^n will be a_0X^n, and, on collecting like powers of X, we shall obtain

$$f(X + h) = a_0X^n + A_1X^{n-1} + \cdots + A_{n-1}X + A_n = 0, \qquad (3)$$

where $A_1, A_2, \cdots; A_n$ are certain coefficients.

Rule 1. — *To obtain the coefficients $A_1, A_2, \cdots, A_n$ in the equation $f(X + h) = 0$, whose roots are those of $f(x) = 0$ each decreased by h, divide $f(x)$ by $(x - h)$; divide the resulting quotient by $(x - h)$; etc. Continue to divide each resulting quotient by $(x - h)$ until n divisions have been performed. The remainder obtained in the first division is A_n; the second remainder is A_{n-1}; $\cdots$; the last remainder is A_1; the last quotient is a_0.*

Illustration. To obtain an equation whose roots are those of $f(x) = 2x^3 - 7x + 6 = 0$ each decreased by 2, we divide by $(x - 2)$, synthetically, as specified in Rule 1:

	2	0	−7	6	|2	
		4	8	2		
First quotient is $2x^2 + 4x + 1$.	2	4	1	8		$A_3 = 8$.
		4	16			
Second quotient is $2x + 8$.	2	8	17			$A_2 = 17$.
		4				
Third quotient is 2.	2	12				$A_1 = 12$.

The desired equation: $2X^3 + 12X^2 + 17X + 8 = 0$.

Proof of Rule 1. — **1.** From (3),

$$f(X + h) = a_0X^n + A_1X^{n-1} + \cdots + A_{n-1}X + A_n. \qquad (4)$$

2. In (4), substitute $X = x - h$; then $f(X + h)$ becomes $f(x)$, and we obtain

$$f(x) = a_0(x - h)^n + A_1(x - h)^{n-1} + \cdots + A_{n-1}(x - h) + A_n. \quad (5)$$

3. From (5), if we divide $f(x)$ by $(x - h)$ the quotient is

$$a_0(x - h)^{n-1} + A_1(x - h)^{n-2} + \cdots + A_{n-2}(x - h) + A_{n-1}, \quad (6)$$

and the remainder is A_n. On dividing (6) by $(x - h)$ the new quotient obtained is

$$a_0(x - h)^{n-2} + A_1(x - h)^{n-3} + \cdots + A_{n-3}(x - h) + A_{n-2},$$

and the remainder is A_{n-1}. Continuing through the n divisions specified in Rule 1, the last quotient is a_0 and the remainders, in the order obtained, are A_n, A_{n-1}, $\cdots$, A_2, A_1. This completes the proof of Rule 1.

EXERCISE 95

Transform each equation into another whose roots are those of the given equation each decreased by the given number h:

1. $2x^3 + 2x^2 - 7x - 72 = 0$: $h = 1$.
2. $2x^3 - 5x^2 + 2x - 4 = 0$: $h = 2$.
3. $x^4 - 3x^2 + 5x + 2 = 0$: $h = .3$.
4. $2x^4 - 4x^3 - 4x + 1 = 0$: $h = .2$.
5. $2x^3 - 3x^2 - 5x + 2 = 0$: $h = -.02$.
6. $x^3 - 4x + 5 = 0$: $h = -.03$.

Obtain an equation each of whose roots is 3 less than a root of the given equation:

7. $4x^3 - 5x^2 - 3x = 4$. **8.** $2x^3 - 4x + 5x^2 = 7$.

Obtain an equation each of whose roots is 2 more than a root of the given equation:

9. $5x^4 - 3x^3 + 1 = 0$. **10.** $2x^4 + 3x^3 - 5x + 7 = 0$.

158. Horner's method for finding irrational roots is illustrated in the following example.

Example 1. — Find the real roots of

$$f(x) = x^3 - 3x^2 - 2x + 5 = 0. \qquad (1)$$

Solution. — **1.** From Step 1 of the solution of Example 1 on page 226, we find that the real roots are approximately -1.3, 1.2, and 3.1.

2. To obtain the root which is between 3 and 4, proceed as follows:

a. *Transform* (1) *to decrease the roots by* 3.

1	−3	−2	5	3
	3	0	−6	
1	0	−2	−1	
	3	9		
1	3	7		
	3			
1	6			

Transformed equation: $f_1(x_1) = x_1^3 + 6x_1^2 + 7x_1 - 1 = 0. \qquad (2)$

A subscript 1 on f_1 and on x_1 serves to distinguish (2) from (1). Since (1) has a root between 3 and 4, hence (2) has a root between 0 and 1. When x_1 is *small*, $(x_1^3 + 6x_1^2)$ is *very* small compared to $(7x_1 - 1)$. Hence, the root of (2) between 0 and 1 is approximately the same as the root of $7x_1 - 1 = 0$, or $x_1 = \frac{1}{7} = .1^+$. To locate this root between successive tenths, we compute values of $f_1(x_1)$ by synthetic division; we find $f_1(.1) = -.239$ and $f_1(.2) = .648$. Hence, (2) has a root between $x_1 = .1$ and $x_1 = .2$.

b. Transform (2) *to decrease the roots by* .1. — The transformed equation, obtained as in Step *a*, is

$$f_2(x_2) = x_2^3 + 6.3x_2^2 + 8.23x_2 - .239 = 0. \qquad (3)$$

Equation 3 has a root between 0 and .1 whose approximate value is obtained from $8.23x_2 - .239 = 0$; $x_2 = .02^+$. To locate this root between successive hundredths, we compute, by synthetic division, $f_2(.02) = -.0719$ and $f_2(.03) = .0136$. Hence, the root is between .02 and .03.

c. Transform (3) *to decrease the roots by* .02. — The transformed equation is

$$f_3(x_3) = x_3^3 + 6.36x_3^2 + 8.4832x_3 - .0719 = 0, \qquad (4)$$

which has a root between 0 and .01 whose approximate value is obtained from $8.4832x_3 - .0719 = 0$; $x_3 = .008^+$.

d. Conclusion. — Each root of (1) is 3.12 greater than a root of (4) because we successively reduced the roots by 3, .1, and .02, or, all together, by 3.12. Since $x_3 = .008$ is approximately a root of (4), hence $x = (3.12 + .008)$, or $x = 3.128$, is approximately a root of (1). This value of the root is *certainly correct to two decimal places*, and is *probably* correct in the third decimal place. If desired, the third decimal place could be checked by locating the root of (4) between successive thousandths.

The essential numerical work performed in obtaining $x = 3.128$ is compactly arranged as follows:

1	− 3	− 2	5	3	
	3	0	− 6		
1	0	− 2	− 1		
	3	9			
1	3	7			
	3				
1	**6**	**7**	**− 1**	.1	$7x_1 - 1 = 0;\ x_1 = .1^+.$
	.1	.61	.761		
1	6.1	7.61	− .239		
	.1	.62			
1	6.2	8.23			
	.1				$8.23\,x_2 - .239 = 0;$
1	**6.3**	**8.23**	**− .239**	.02	$x_2 = .02^+.$
	.02	.1264	+ .1671		
1	6.32	8.3564	− .0719		
	.02	.1268			
1	6.34	8.4832			
	.02				$8.4832\,x_3 - .0719 = 0;$
1	**6.36**	**8.4832**	**− .0719**		$x_3 = .008^+.$

3. By the method of Step 2, we find $x = 1.202$ as the second positive root.

4. To find the negative root of $f(x) = 0$, consider the equation

$$f(-x) = -x^3 - 3x^2 + 2x + 5 = 0.$$

Since $f(x) = 0$ has a root near $x = -1.3$, hence $f(-x) = 0$ has a root near $x = 1.3$. By the method of Step 2, this root of $f(-x) = 0$ is found to be $x = 1.330$. Hence, $x = -1.330$ is the negative root of $f(x) = 0$.

Horner's method for finding a positive irrational root:

1. *Locate the root between successive integers. The smaller integer is the integral part of the root.*

2. *Transform* $f(x) = 0$ *into an equation* $f_1(x_1) = 0$ *whose roots are those of* $f(x) = 0$ *decreased by the smaller of the integers of Step* 1. *Then,* $f_1(x_1) = 0$ *has a root between* 0 *and* 1. *Locate this root between successive tenths. The smaller tenth is the tenths part of the root.*

3. *Transform* $f_1(x_1) = 0$ *into an equation* $f_2(x_2) = 0$ *whose roots are those of* $f_1(x_1) = 0$ *decreased by the smaller of the successive tenths found in Step* 2. *Then,* $f_2(x_2)$ *has a root between* 0 *and* .1; *locate this root between successive hundredths. The smaller hundredth is the hundredths part of the root.*

4. *Continue this process, to obtain any desired accuracy. Each step yields, accurately, one more decimal place of the root.*

159. Outline for finding the real roots of an equation:

1. *If the coefficients are rational, find all rational roots. Let* $f(x) = 0$ *be the depressed equation after all rational roots have been removed; then, any real root of* $f(x) = 0$ *is irrational.*

2. *Obtain limits for the real roots of* $f(x) = 0$ *by the method of Section* 152. *Compute* $f(x)$ *for integral values of* x *between these limits, and obtain a rough graph of* $f(x)$.

3. *Read the approximate value of each real root from the graph.*

4. *By successive graphs, or by Horner's method, obtain each irrational root in turn to the desired number of decimal places.*

NOTE 1. — It should be recognized that Horner's method is applicable only to integral rational equations, whereas the method of Section 156 applies to transcendental equations, or irrational equations, as well as it does to integral rational equations.

NOTE 2. — If all roots of $f(x) = 0$ are found to be real, the roots obtained can be checked by use of the following theorem, which will be proved in a later section: *In an equation* $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, *the sum of the roots equals* $-\frac{a_1}{a_0}$. Thus, in Example 1, Section 158,

$-\frac{a_1}{a_0} = 3$, and the sum of the roots found is $3.128 + 1.202 - 1.330 = 4.330 - 1.330 = 3$.

NOTE 3. — If Descartes' rule of signs has been studied, it should be applied to obtain information about the roots as the first step in the solution of an equation.

EXERCISE 96 *

Find the specified roots, accurately to three decimal places:

1. The root of $x^3 + x^2 - x - 99 = 0$ between 4 and 5.

2. The root of $2x^3 + 5x^2 - x = 5$ between -2 and -1.

3. The root of $x^3 - 2x^2 - x + 1 = 0$ between -1 and 0.

4. The root of $x^4 + x^3 + x^2 - 2x - 6 = 0$ between 1 and 2.

5. The two roots of $5x^3 - 9x^2 - 4x + 9 = 0$ between 1 and 2.

HINT. — Substitute $x = 1$, $x = 1.1$, $x = 1.2$, $\cdots$, in succession, until both of the roots are located between successive tenths.

6. The two roots of $x^3 - 7x + 7 = 0$ between 1 and 2.

Find all real roots of each equation:

7. $x^4 + 3x^3 + x^2 - 2x - 1 = 0$.

8. $x^3 - 2x^2 - 7x - 1 = 0$.

9. $x^4 + 2x^3 - 5 = 0$.

10. $x^3 - 5x^2 + 2x + 6 = 0$.

11. $2x^4 - 3x^3 + 5x^2 - 4x + 1 = 0$.

12. $3x^3 - 16x^2 + 25x - 11 = 0$.

13. $x^4 - 6x^3 + 6x^2 + 8x + 1 = 0$.

14. $x^3 - 6x^2 + 15x - 19 = 0$.

15. $x^4 - 7.3 = 0$.

16. $x^5 + 25 = 0$.

17. $x^3 - x^2 - 15x + 28 = 0$.

18. $x^3 - 9x^2 + 23x - 16 = 0$.

* Contrary to custom, answers are given for problems 2, 4, and 6.

Find the indicated principal roots, correct to three decimal places:

19. $\sqrt[4]{7}$. **20.** $\sqrt[3]{246}$. **21.** $\sqrt[3]{-53}$. **22.** $\sqrt[5]{-135}$.

HINT. — The positive root of $x^4 - 7 = 0$ is the value of $\sqrt[4]{7}$.

The following equations have no rational roots. Find their real roots, correct to two decimal places:

23. $2.1\,x^3 - 3.13\,x^2 + .859 = 0$. **25.** $x^3 + 12\,x - 300 = 0$.

24. $.235\,x^3 - .683\,x^2 + .183 = 0$. **26.** $x^4 - 350\,x - 1200 = 0$.

In the following systems, find all real solutions (a) graphically; (b) algebraically:

27. $\begin{cases} y = x^2 - 2, \\ y^2 - 6\,y - x + 8 = 0. \end{cases}$ **28.** $\begin{cases} y = x^2 + 2\,x + 1, \\ 4\,x^2 + 9\,y^2 = 36. \end{cases}$

29. An open box is to be made from a rectangular piece of cardboard, 15″ long and 10″ wide, by cutting equal squares from the corners and turning up the sides. Find the side of these squares if the box is to contain 120 cubic inches.

30. The edges of a rectangular box are 3′, 4′, and 5′ long. To double the volume, each dimension is increased by the same amount. Find the new dimensions.

31. A spherical shell has an inner radius of 8″; the volume of the solid part is 1000 cubic inches. Find the thickness of the shell.

The depth d to which a solid floating sphere will sink in water is a positive root of the equation $d^3 - 3\,rd^2 + 4\,r^3 s = 0$, *where r is the radius of the sphere and s is the specific gravity of the substance composing the sphere. Find d for each sphere below:*

32. Cork sphere: radius = 2′; specific gravity = .21.

33. Wooden sphere: radius = 4′; specific gravity = .71.

SUPPLEMENTARY MATERIAL

160. Coefficients in terms of the roots.

Illustration. Let r_1, r_2, and r_3 be the roots of

$$x^3 + b_1x^2 + b_2x + b_3 = 0. \quad \cdots \quad (1)$$

From Section 143, $x^3 + b_1x^2 + b_2x + b_3 \equiv (x - r_1)(x - r_2)(x - r_3)$

$$\equiv x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3. \quad (2)$$

From Corollary 1, page 210, it follows that

$$b_1 = -(r_1 + r_2 + r_3);\ b_2 = (r_1r_2 + r_1r_3 + r_2r_3);\ b_3 = -r_1r_2r_3. \quad (3)$$

Similarly, we find that, if $r_1, r_2, \cdots, r_n$ are the roots of

$$x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n = 0, \quad (4)$$

the following equations are true:

$$\left.\begin{aligned} b_1 &= -(r_1 + r_2 + \cdots + r_n),\\ b_2 &= (r_1r_2 + r_1r_3 + \cdots + r_1r_n + r_2r_3 + \cdots + r_{n-1}r_n),\\ b_3 &= -(r_1r_2r_3 + r_1r_2r_4 + \cdots),\\ &\cdots\cdots\cdots\cdots\cdots\\ b_n &= (-1)^n r_1r_2r_3 \cdots r_n. \end{aligned}\right\} \quad (5)$$

Or, $b_1 = -$(*the sum of all of the roots*),
$b_2 = +$(*the sum of the products of the roots, two at a time*),
$b_3 = -$(*the sum of the products of the roots, three at a time*),
. . . . . .
$b_n = (-1)^n \cdot$ (*the product of the roots*).

NOTE. — If we divide both sides of

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0 \quad (6)$$

by a_0, we obtain $x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \cdots + \frac{a_n}{a_0} = 0.$

Hence, from (5), if $(r_1, r_2, \cdots, r_n)$ are the roots of (6), then

$$\frac{a_1}{a_0} = -(r_1 + r_2 + \cdots + r_n);\ \frac{a_2}{a_0} = +(r_1r_2 + r_1r_3 + \cdots);\ \text{etc.}$$

EXERCISE 97

1. By the method used in obtaining equations (3), above, derive expressions (5) for the case of $x^4 + b_1x^3 + b_2x^2 + b_3x + b_4 = 0$.

In the remaining problems, x is the unknown.

2. Find the third root of $2x^3 - 8x^2 + bx + d = 0$, given that 3 and -7 are two of the roots.

3. Find all of the roots of $3x^3 - 9x^2 + cx + f = 0$, given that one root is 5 and the difference of the other roots is 4.

4. Solve $x^3 - 4x^2 - 9x + h = 0$, given that one root is the negative of another.

5. Solve $x^3 + 6x^2 - 4x + h = 0$, given that the roots, in some order, form an arithmetical progression.

HINT. — Let the roots be a, $a + d$, and $a + 2d$.

6. Solve $x^4 - 2x^3 - 11x^2 + kx + 36 = 0$, given that it has two double roots.

7. Solve $x^3 + 7x^2 + 14x + k = 0$, given that the roots, in some order, form a geometrical progression.

8. Prove that, if $x^3 + qx^2 + rx + s = 0$ has one root the negative of another, then $s = qr$.

9. If the roots of $x^3 + bx^2 + cx + d = 0$, taken in some order form a geometrical progression, prove that $c^3 - b^3d = 0$.

161. Algebraic solution* of equations. — In elementary algebra we obtained formulas for the solutions of the general linear equation $ax + b = 0$, and of the general quadratic equation $ax^2 + bx + c = 0$. We shall now obtain formulas for the solutions of the general cubic and quartic equations and thus complete the proof of the following theorem.

Theorem I. — *If $f(x) = 0$ represents the general equation of the first, or second, or third, or fourth degree, then the roots of $f(x) = 0$ can be expressed in terms of the coefficients of $f(x)$ by means of formulas involving a finite number of the following operations, and no others: addition, subtraction, multiplication, division, and the extraction of roots.*

NOTE. — It was proved in 1824 by the Norwegian mathematician, NIELS HENRIK ABEL (1802–1829), that the preceding statement *cannot be made* for the general equation $f(x) = 0$ of degree 5, or more.

NOTE. — Usually, real roots of particular cubic or quartic equations are determined more conveniently by Horner's method than by the formulas of the following sections.

*Section 139 is a prerequisite for the remainder of the chapter.

162. Tartaglia's solution of the general cubic equation,

$$x^3 + bx^2 + cx + d = 0. \tag{1}$$

Solution. — In (1), place $$x = y - \frac{b}{3}. \tag{2}$$

One obtains $$y^3 + py + q = 0 \tag{3}$$

where $p = c - \frac{b^2}{3}$, and $q = d - \frac{bc}{3} + \frac{2\,b^3}{27}$.

Equation 3 is called the **reduced cubic**; the advantage of (3) is that it contains no term in y^2. In (3), place

$$y = z - \frac{p}{3\,z}. \tag{4}$$

Then, $$z^3 - \frac{p^3}{27\,z^3} + q = 0$$

or, $$z^6 + qz^3 - \frac{p^3}{27} = 0. \tag{5}$$

Since (5) is a quadratic in z^3, we can obtain the values of z^3 by use of the quadratic formula. If we let $R = \frac{p^3}{27} + \frac{q^2}{4}$, then

$$z^3 = -\frac{q}{2} + \sqrt{R}, \tag{6}$$

and $$z^3 = -\frac{q}{2} - \sqrt{R}. \tag{7}$$

Equation 6 has three solutions, z_1, z_2, and z_3, which are the three cube roots of $\left(-\frac{q}{2} + \sqrt{R}\right)$. From (4), the corresponding values of y are

$$y_1 = z_1 - \frac{p}{3\,z_1},\ y_2 = z_2 - \frac{p}{3\,z_2},\ \text{and}\ y_3 = z_3 - \frac{p}{3\,z_3}.$$

Therefore, from (2), the roots of equation 1 are

$$x_1 = z_1 - \frac{p}{3\,z_1} - \frac{b}{3},\ x_2 = z_2 - \frac{p}{3\,z_2} - \frac{b}{3},\ x_3 = z_3 - \frac{p}{3\,z_3} - \frac{b}{3}. \tag{8}$$

It can be proved that if in place of (z_1, z_2, z_3) we had used the solutions (z_4, z_5, z_6) of (7), the three values of x obtained would be the same as those given in (8).

NOTE. — The quantities p, q, and R, introduced in the solution, are rational functions of b, c, and d, the coefficients in (1). Hence, we should think of z_1, z_2, and z_3 as the indicated cube roots of a certain function of b, c, and d. Therefore, in (8) we have formulas for x_1, x_2, and x_3 in terms of the coefficients (b, c, d). Moreover, these formulas involve only a certain number of the following types of operations: addition, subtraction, multiplication, division, and the extraction of square and cube roots.

NOTE. — The method of solution presented in this section was discovered independently by SCIPIONE DEL FERRO (1496–1526) and TARTAGLIA (1500–1557). The method was first published in 1545 by CARDANO (1501–1576) in a famous treatise called *Ars Magna*, although Cardano had obtained the method from Tartaglia under a promise of secrecy. The expressions in (8), when more explicitly written, are known as *Cardan's formulas* for the solutions of a cubic.

163. Ferrari's solution of the general quartic

$$x^4 + bx^3 + cx^2 + dx + e = 0. \tag{1}$$

Solution. — From (1), $x^4 + bx^3 = -cx^2 - dx - e.$ (2)

We shall determine what expression should be added to both sides of (2) to make each side a perfect square. The left member already contains the first two terms of $(x^2 + \frac{1}{2}bx)^2$. Hence, on adding $\frac{1}{4}b^2x^2$ to both sides of (2), we obtain

$$(x^2 + \tfrac{1}{2}bx)^2 = (\tfrac{1}{4}b^2 - c)x^2 - dx - e. \tag{3}$$

Let y be a number to be specified later. On adding $y(x^2 + \frac{1}{2}bx) + \frac{1}{4}y^2$ to both sides of (3), we obtain

$$(x^2 + \tfrac{1}{2}bx + \tfrac{1}{2}y)^2 = (\tfrac{1}{4}b^2 - c + y)x^2 + (\tfrac{1}{2}by - d)x + (\tfrac{1}{4}y^2 - e). \tag{4}$$

The right side of (4) is a quadratic in x which will be a perfect square if its discriminant is zero, that is, if

$$(\tfrac{1}{2}by - d)^2 - 4(\tfrac{1}{4}b^2 - c + y)(\tfrac{1}{4}y^2 - e) = 0. \tag{5}$$

On simplifying (5) we obtain

$$y^3 - cy^2 + (bd - 4e)y - b^2e + 4ce - d^2 = 0. \tag{6}$$

Equation 6 is called the **resolvent cubic** for equation 1; (6) can be solved by the method of Section 162.

Let $y = S$ be any root of (6), and substitute $y = S$ in (4). The right side of (4) becomes the square of some linear function, $(hx + k)$, and (4) becomes

$$(x^2 + \tfrac{1}{2}bx + \tfrac{1}{2}S)^2 = (hx + k)^2. \tag{7}$$

From (7) $$x^2 + \tfrac{1}{2}bx + \tfrac{1}{2}S = hx + k, \tag{8}$$

or $$x^2 + \tfrac{1}{2}bx + \tfrac{1}{2}S = -(hx + k). \tag{9}$$

Each of (8) and (9) can be solved by use of the quadratic formula. The four values of x obtained* on solving (8) and (9) are the roots of (1).

NOTE. — The preceding method for solving a quartic was invented by FERRARI (1522–1565) and was first published by Cardano in his *Ars Magna* in 1545.

NOTE. — By use of the formulas of Section 162, we could write a formula for the root S, of the resolvent cubic, which we used in the preceding solution. This formula would involve the coefficients (b, c, d, e) of (1). Using this formula for S, we could obtain formulas for h and k, in terms of (b, c, d, e); and, solving (8) and (9) in terms of h, k, and S, we could obtain formulas for the four roots of (1), in terms of (b, c, d, e). These formulas would be very complicated, and it is merely of theoretical interest to notice that they could actually be obtained.

EXERCISE 98

Solve each equation by taking the steps performed in arriving at Cardan's formulas, but do not use the formulas themselves:

1. $x^3 - 3x^2 + 48x + 52 = 0$. **3.** $x^3 - 4x^2 + 3x + 2 = 0$.

2. $x^3 + 6x^2 + 27x - 86 = 0$. **4.** $x^3 - 6x - 4 = 0$.

Solve by Ferrari's method:

5. $x^4 + x^3 - 4x^2 - 2x + 4 = 0$.

HINT. — The resolvent cubic has a rational negative root.

6. $x^4 - 60x^2 + 80x + 384 = 0$.

7. $x^4 - x^3 - 6x^2 + 2x + 4 = 0$.

8. $x^4 + 4x^3 - 4x^2 - 37x - 42 = 0$.

*It can be proved that, if any root of (6) other than $y = S$ were used in (7), the same four values of x would result.

CHAPTER XXI

LOGARITHMS

164. Logarithms are exponents, and by use of them we can simplify certain arithmetical computations.

Let a be positive and not equal to 1, and let N be any positive number. Then, **the logarithm of N to the base a is the exponent of the power to which a must be raised to obtain N.**

To abbreviate "*the logarithm of N to the base a*" we write "$\log_a N$" which, however, is read without abbreviation. By the definition of $\log_a N$, if either of the equations

$$N = a^x \text{ and } x = \log_a N \tag{1}$$

is true the other is also. It is convenient to remember that

$$N = a^{\log_a N} \tag{2}$$

Thus, by use of equations 1, if $\log_4 N = 5$, then $N = 4^5$; if $N = 7^3$, then $\log_7 N = 3$; if $\log_a 32 = 5$, then $a^5 = 32$, or $a = 2$.

EXERCISE 99

1. By use of equations 1, complete each column:

SINCE	$9 = 3^2$,	$625 = 5^4$,	$100 = 10^2$,	$\frac{1}{27} = 3^{-3}$,
THEN	$\log_3 9 =$	$\log_5 625 =$	$\log_{10} 100 =$	$\log_3 \frac{1}{27} =$

2. Find the number N corresponding to each logarithm:

$\log_5 N = 3$.	$\log_2 N = 1$.	$\log_{10} N = 0$.	$\log_a N = -\frac{1}{3}$.
$\log_{49} N = \frac{1}{2}$.	$\log_5 N = 1$.	$\log_{10} N = -2$.	$\log_{10} N = 5$.
$\log_{16} N = \frac{1}{4}$.	$\log_{81} N = -\frac{1}{2}$.	$\log_3 N = 4$.	$\log_a N = 6$.
$\log_9 N = -2$.	$\log_4 N = 2$.	$\log_8 N = \frac{1}{3}$.	$\log_{10} N = -1$.
$\log_7 N = 1$.	$\log_4 N = \frac{1}{2}$.	$\log_a N = 0$.	$\log_a N = -4$.

3. Find $\log_6 216$; $\log_2 \frac{1}{32}$.

Solution. — Express each number as a power of the specified base.

$216 = 6^3$; hence, $\log_6 216 = 3$. $\frac{1}{32} = \frac{1}{2^5} = 2^{-5}$; hence, $\log_2 \frac{1}{32} = -5$.

4. Find the following logarithms:

$\log_7 49$	$\log_4 2$	$\log_5 25$	$\log_{100} 10$
$\log_9 81$	$\log_3 \frac{1}{9}$	$\log_a 1$	$\log_{10} \frac{1}{100}$
$\log_{25} 5$	$\log_2 16$	$\log_{10} 1000$	$\log_{10} \frac{1}{10}$
$\log_3 \frac{1}{3}$	$\log_a a$	$\log_3 \frac{1}{81}$	$\log_{10} \frac{1}{10000}$
$\log_6 36$	$\log_8 2$	$\log_{10} 10{,}000$	$\log_{10} .0001$
$\log_{10} 10$	$\log_2 \frac{1}{64}$	$\log_{16} 4$	$\log_{10} .00001$

5. Find the unknown base in each case below:

$\log_a 16 = 2$.	$\log_a 2 = \frac{1}{3}$.	$\log_a 625 = 4$.	$\log_a 1000 = 3$.
$\log_b 125 = 3$.	$\log_a 4 = 2$.	$\log_b 25 = 2$.	$\log_a .001 = -3$.
$\log_a 10 = \frac{1}{2}$.	$\log_d \frac{1}{25} = -1$.	$\log_c 81 = 4$.	$\log_d .0001 = -2$.

NOTE. — It is impossible to use $a = 1$ as the base for a system of logarithms, because every power of 1 is 1 and hence no number except 1 could have a logarithm to the base 1. If a negative base were used, the logarithms of most numbers would be imaginary numbers. Hence, in the definition of a logarithm, we assumed that $a > 0$ and $a \neq 1$.

165. We shall assume the truth of the following statements, whose proofs are given in advanced mathematics:

1. *The laws of exponents apply in case the exponents involved are any real numbers, rational or irrational.*

2. *If $a > 0$ and $\neq 1$, then, for every positive number N, there exists one and only one real number x such that $N = a^x$.*

NOTE. — Assumption 2 states that *every positive number N has one and only one real logarithm to the base a.* In advanced mathematics it is shown that, if N is negative, $\log_a N$ can be defined as an imaginary number.

In this book, if we talk of the logarithm of a number N, it will be implied that N is a positive number.

166. Useful properties of logarithms.

I. *The logarithm of a product equals the sum of the logarithms of its factors.*

Proof. — Consider a product MN. We wish to show that

$$\log_a MN = \log_a M + \log_a N. \tag{1}$$

1. Let $x = \log_a M$ and $y = \log_a N$. Then,

2. $M = a^x$ and $N = a^y$. (Def. of a logarithm)

3. Hence, $MN = a^x a^y = a^{x+y}$. (Law of exponents)

4. Since $MN = a^{x+y}$, then $\log_a MN = x + y$. (Def. of a log.)

Or, substituting $x = \log_a M$ and $y = \log_a N$,

$$\log_a MN = \log_a M + \log_a N.$$

Similarly we could prove that

$$\log_a MNP = \log_a M + \log_a N + \log_a P,$$

etc., for any number of factors.

Illustration. $\log_{10}(8967)(5632) = \log_{10} 8967 + \log_{10} 5632$.

II. *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor;* or

$$\log_a \frac{M}{N} = \log_a M - \log_a N. \tag{2}$$

Proof. — **1.** Let $x = \log_a M$ and $y = \log_a N$. Then

2. $M = a^x$ and $N = a^y$. (Def. of a logarithm)

3. Hence $\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y}$. (Law of exponents)

4. Since $\frac{M}{N} = a^{x-y}$, then $\log_a \frac{M}{N} = x - y$. (Def. of a log.)

Or, $\log_a \frac{M}{N} = \log_a M - \log_a N$. (Subst. $x = \log_a M$; $y = \log_a N$)

Illustration. $\log_{10} \frac{89}{57} = \log_{10} 89 - \log_{10} 57$.

III. *The logarithm of the kth power of N equals k times the logarithm of N;* or

$$\log_a N^k = k \log_a N. \tag{3}$$

Proof. — **1.** Let $x = \log_a N$; then $N = a^x$. (Def. of a log.)

2. Hence, $N^k = (a^x)^k = a^{kx}$. (Law of exponents)

3. Therefore, $\log_a N^k = kx = k \log_a N$. (Subst. $x = \log_a N$)

Illustration. $\log_{10} 75^{9.7} = 9.7 \log_{10} 75$.

Since $N^{\frac{1}{h}} = \sqrt[h]{N}$, we obtain, from equation 3 with $k = \frac{1}{h}$,

$$\log_a \sqrt[h]{N} = \frac{1}{h} \log_a N. \tag{4}$$

Illustration. $\log_{10} \sqrt[3]{532} = \frac{1}{3} \log_{10} 532$.

EXERCISE 100

Prove each statement by the method used for Properties I *and* II:

1. $\log_a MNP = \log_a M + \log_a N + \log_a P$.

2. $\log_a (MP) \div Q = \log_a M + \log_a P - \log_a Q$.

167. Logarithms to the base 10 are called **common**, or **Briggs** logarithms. Hereafter, unless otherwise stated, *when we mention a logarithm we shall mean a common logarithm.* Instead of writing $\log_{10} N$ for the common logarithm of N, we shall omit the 10 and write merely $\log N$. Thus, $\log 3$ means $\log_{10} 3$.

NOTE. — Common logarithms are the most convenient logarithms for computational purposes. The only other variety used appreciably is the system of ***natural***, or ***Naperian***, logarithms, in which the base is a certain irrational number denoted by e where $e = 2.71828$, approximately. Naperian logarithms are useful for theoretical reasons.

HISTORICAL NOTE. — Logarithms were invented by a Scotchman, JOHN NAPIER, Baron of Merchiston (1550–1617). His original logarithms were not the same as those now called Naperian logarithms. Common logarithms were invented by an Englishman, HENRY BRIGGS (1556–1631), who was aided in his invention by Napier. Briggs published the first table of common logarithms in 1624.

168. Characteristic and mantissa. — If N is an integral power of 10, $\log N$ is an integer. Otherwise, $\log N$ is a number which we can express as a decimal fraction. Every number, and hence in particular *every logarithm, can be written as the sum of an integer and a positive decimal,* < 1:

$$\log N = (\textit{an integer}) + (\textit{a positive decimal}, < 1). \qquad (1)$$

After $\log N$ has been written in this form, we call the integer the **characteristic** of $\log N$, and the positive decimal, < 1, the **mantissa** of $\log N$. Thus,

$$\log N = \text{characteristic} + \text{mantissa}. \qquad (2)$$

Given that $\log .0385 = -1.41454 = -2 + .58546$, we see that -2 is its characteristic and .58546 is its mantissa.

EXERCISE 101

Find the characteristic and the mantissa of each given logarithm:

1. $\log 8000 = 3.90309$.
2. $\log .75 = -.12494$.
3. $\log .04 = -1.39794$.
4. $\log 4.8 = 0.68124$.
5. $\log 213.5 = 2.32940$.
6. $\log 25 = 1.39794$.
7. $\log .061 = -1.21467$.
8. $\log .0003 = -3.52288$.
9. $\log .005 = -2.30103$.
10. $\log .03 = -1.52288$.

169. Rules for finding the characteristic. — The student should verify the following logarithms, by use of the definition of a logarithm, recalling that the base is 10.

Number	.0001	.001	.01	.1	1	10	100	1000	10,000	100,000
Logarithm	-4	-3	-2	-1	0	1	2	3	4	5

So far as integral powers of 10 are concerned, we see that *if one number is less than another, the logarithm of the first is less than the logarithm of the second.*

Thus, $.001 < 100$; and $\log .001 < \log 100$, or $-3 < 2$.

In general, let us assume, without proof, the fact that **whenever $a < b$, then $\log a < \log b$.** Hence,

$$\text{if } M < N < P, \text{ then } \log M < \log N < \log P. \qquad (1)$$

By use of statement 1 we can determine the characteristic of the logarithm of any number.

Illustration 1. We find the characteristic of log 375 as follows:

Since $100 < 375 < 1000$, hence $\log 100 < \log 375 < \log 1000$.

Therefore, $2 < \log 375 < 3$; hence

$$\log 375 = 2 + (a \textit{ positive decimal}, < 1).$$

Therefore, the characteristic of log 375 is 2.

Illustration 2. We find the characteristic of log .053 as follows:

Since $.01 < .053 < .1$, hence $\log .01 < \log .053 < \log .1$.

Therefore, $-2 < \log .053 < -1$; hence,

$$\log .053 = -2 + (a \textit{ positive decimal}, < 1).$$

Therefore, the characteristic of log .053 is -2.

These examples illustrate the following rules:

Rule 1. — When $N > 1$, the characteristic of log N is a positive integer which is one less than the number of digits in N to the left of the decimal point.

Rule 2. — When $N < 1$, the characteristic of log N is a negative integer. If the first significant digit of N is in the kth decimal place, then $-k$ is the characteristic of log N.

By use of Rules 1 and 2, we determine the characteristic of log N by merely inspecting N.

Thus, the characteristic of log 7356.5 is $(4 - 1)$, or 3; the characteristic of log .00039 is -4, because "3" is in the 4th place.

EXERCISE 102

By the method used in Illustrations 1 *and* 2 *of Section* 169, *find the characteristic of the logarithm of each number:*

1. 2957. **2.** .0083. **3.** .6573. **4.** 1.895. **5.** 23. **6.** 8.

7. 57,850. **13.** .0000008. **19.** .995.

8. 139.4. **14.** 5697.2. **20.** .0000000051.

9. 31.25. **15.** .00192. **21.** 1,139,000.

10. .1398. **16.** .000056. **22.** 425.7.

11. .0389. **17.** 5.139. **23.** 489,000.

12. .00089. **18.** 11.35. **24.** .000315.

25. What is the characteristic of log 10,000; of log .001?

26. Write an illustration of a number N for which the characteristic of log N is 4; 2; 0; -3; -2; 3.

170. Property of the mantissa. — Two numbers such as .00038037 and 3,803,700 are said to have the same **significant** digits because, aside from the initial zeros of .00038037 and the final zeros of 3,803,700, the same sequence of digits (3 8 0 3 7) occurs in each number.

Illustration 1. Given that log 3.8037 = .58021, then, by use of Properties I and II of Section 166, we obtain the following results:

$$\log 380.37 = \log 100(3.8037) = \log 100 + \log 3.8037 = 2 + .58021;$$
$$\log 38.037 = \log 10(3.8037) = \log 10 + \log 3.8037 = 1 + .58021;$$
$$\log 3.8037 = .58021 = 0 + .58021;$$
$$\log .38037 = \log \frac{3.8037}{10} = \log 3.8037 - \log 10 = -1 + .58021;$$
$$\log .038037 = \log \frac{3.8037}{100} = \log 3.8037 - \log 100 = -2 + .58021;$$

We see that .58021 is the mantissa of each logarithm. Similarly, if N is *any* number whose significant digits are (3 8 0 3 7), then N equals 3.8037 multiplied by a positive or negative integral power of 10. Hence,

$$\log N = \log 3.8037 + (an\ integer) = .58021 + (an\ integer),$$

and therefore .58021 is the mantissa of log N.

In Illustration 1, if we had considered numbers with *any* given sequence of significant digits, similar results would have been obtained. Therefore, we may state the following rule.

Rule 1. — The mantissa of log N depends only on the sequence of significant digits in N, and, hence, if two numbers differ only in the position of the decimal point, their logarithms have the same mantissa:

Because of Rule 1, we speak of a mantissa as *corresponding to a given sequence of digits*, without thinking of any decimal point associated with the digits. Thus, we say that .58021 is the *mantissa for the digits* 38037, meaning that .58021 is the mantissa of the logarithm of any number whose significant digits form the sequence 38037. Similarly, if .76664 is the mantissa for the digits 5843, then $\log 584.3 = 2.76664$; $\log .0005843 = -4 + .76664$.

Mantissas can be computed by the methods of advanced mathematics. We find the computed mantissas in *tables of logarithms* which may also be called *tables of mantissas.* Except in special cases, mantissas are infinite decimal fractions.

Thus, the mantissa for 10705 is .029586671630457, to fifteen decimal places. In a *five-place* table of logarithms this mantissa would be recorded *correct to five decimal places*, giving .02959. In an *eight-place* table, it would be recorded as .02958667, *correct to eight places.*

There is a *four-place* table of mantissas on page 248. A decimal point is understood in front of each mantissa.

Illustrations. To find the mantissa for 356, search for 35 in the column headed N; go out in the row of 35 to the column headed 6 at the top of the page. The entry .5514 is the mantissa for 356. The student should now verify the following mantissas.

FOR THE DIGITS	363	478	866	139	299	962
THE MANT. IS	.5599	.6794	.9375	.1430	.4757	.9832

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

171. Four-place logarithms of numbers with three significant digits.

Example 1. — Find log 38,700; log .00843.

Solution. — 1. By Rule 1, Section 169, the characteristic of log 38,700 is 4; from the table of mantissas, the mantissa for 387 is .5877. Hence, log 38,700 = 4 + .5877 = 4.5877.

2. By Rule 2, Section 169, the characteristic of log .00843 is − 3. The mantissa for 843 is .9258. Hence, log .00843 = − 3 + .9258.

NOTE. — We write log .00843 = − 3 + .9258 instead of performing the addition and writing log .00843 = − 2.0742, because in the form (− 3 + .9258) the characteristic and mantissa are in plain view.

EXERCISE 103

1. By use of the table of mantissas verify the logarithms:

N	log N	N	log N
3520	3.5465	831,000	5.9196
.0183	− 2 + .2625	.00563	− 3 + .7505
98.7	1.9943	4.56	0.6590

2. Find the logarithm of each of the following numbers:

53.2	.0163	2990	.0001	20,000	.087	999,000	5.98
6.97	.00726	.0342	.00099	13,200	3.01	.00009	6.26

3. Given that log 356.27 = 2.55178, what is the value of log 35,627; log .035627; log .000035627; log 3.5627?

NOTE. — When N is a number between 1 and 10, the characteristic of log N is 0, and hence log N equals its mantissa. Thus, from the table of mantissas, log 3.57 = 0.5527. Hence, a table of mantissas is a table of the *actual logarithms* of numbers between 1 and 10.

172. Four-place logarithms of numbers with four significant digits.

Example 1. — Find log 13.86.

Solution. — Notice that $13.80 < 13.86 < 13.90$; hence, $\log 13.80 < \log 13.86 < \log 13.90$. In the adjoining table we give $\log 13.80$ and $\log 13.90$ as read from the table of mantissas. An increase of .10 in N, $(13.90 - 13.80 = .10)$, causes an increase of .0031, $(1.1430 - 1.1399 = .0031)$, in $\log N$. Since $\frac{.06}{.10} = .6$, we *assume* that a change of .06 in N, $(13.86 - 13.80 = .06)$, will cause .6 as much of an increase in $\log N$, or an increase of $.6(.0031)$, or .00186, or .0019 approximately. Hence,

N	$\log N$
13.80	1.1399
13.86	?
13.90	1.1430

$$\log 13.86 = 1.1399 + .0019 = 1.1418.$$

Note 1. — In Example 1, .0031 is called the **tabular difference**; .0019 is called the **proportional part** of the tabular difference. In the solution, $.6(.0031) = .00186$, but, since the mantissas in the table are given to only four decimal places, we reduce .00186 to .0019, the nearest number of four decimal places.

Note 2. — The solution of Example 1 illustrates the method of **interpolation** by the *principle of proportional parts*, which states that, for small changes in N, the corresponding changes in $\log N$ will be proportional to the changes in N. In advanced mathematics, it is proved that, by interpolation as in Example 1, we obtain logarithms which, with rare exceptions, are accurate to four decimal places.

Example 2. — Find $\log .0001785$.

Solution. — Mantissa for 1780 is 2504 } Tabular diff. is 25.
Mantissa for 1790 is 2529 } $.5(25) = 12.5$.
Mantissa for 1785 is $.2504 + .0012 = .2516$.

(Rule 2, Section 169) Characteristic of $\log .0001785$ is -4.

$$\therefore \log .0001785 = -4 + .2516.$$

Note 3. — In Example 2, $.5(25) = 12.5$; with equal justification, we could call this either 12 or 13. In this book, whenever such an ambiguity is met *we agree to choose the even number.*

173. For later convenience, we shall hereafter write negative characteristics in a new way. Thus, in $\log .0001785 = (-4 + .2516)$, change -4 to $(6 - 10)$, which it equals. Then,

$$\log .0001785 = -4 + .2516 = (6 - 10) + .2516 = 6.2516 - 10.$$

Recognize clearly that $\log .0001785 = -3.7484$ *and that we have merely introduced a new way for writing this negative number* in such a manner that the characteristic, -4 or $(6-10)$, and the mantissa .2516 will remain in evidence:

$$\begin{array}{rr} -4.0000 & -10.0000 \\ +\ .2516 & +\ 6.2516 \\ -4+.2516 = -3.7484 & -\ 3.7484 = 6.2516 - 10 \end{array}$$

To write a negative characteristic $-k$ in the new form, *we change* $-k$ *to* $[(10-k)-10]$. For instance,

$$\log .01785 = -2 + .2516 = 8.2516 - 10;$$
$$\log .1785 = -1 + .2516 = 9.2516 - 10.$$

NOTE 1. — In this book, to avoid unnecessary arithmetic in interpolation when a number N has more than four significant digits, **we agree to change N to the nearest number with four significant digits, before finding the four-place logarithm of N.** The point of the rule* is illustrated by the fact that 473.76 and 473.80 have the same four-place logarithm.

EXERCISE 104

1. What are the characteristics of the following logarithms: $9.8542 - 10$; $7.3156 - 10$; $8.4216 - 10$.

2. Write the following logarithms in the form of Section 173: $-3 + .4821$; $-5 + .356$; $-1 + .0312$; $-2 + .5829$.

3. Write the following logarithms as pure negative numbers: $-3 + .5678$; $7.5678 - 10$; $-5 + .7654$; $8.4217 - 10$.

4. Read Note 1 in Section 173, and verify the following logarithms:

$\log 256.3 = 2.4087$	$\log .00047985 = 6.6810 - 10$
$\log 13.79 = 1.1396$	$\log .0031529 = 7.4987 - 10$
$\log .023556 = 8.3722 - 10$	$\log 9.347 = 0.9707$
$\log 28{,}432 = 4.4537$	$\log 1.1083 = 0.0445$

* Some computers make an exception to this rule in case the first significant digit of N is < 4; in such cases they change N to the nearest number of five significant digits.

5. Find the logarithms of the following numbers:

18.6	.009567	.3146	.061932	151.1
5322	5.632	83,200	.48738	63.19
67.596	1.2786	113.4	384.218	.0000003172
.031584	.03264	.92	527.9	840,000

174. To find a number when its four-place logarithm is given.

Example 1. — Find N if $\log N = 7.6064 - 10$.

Solution. — 1. In the table of mantissas, .6064 is found as the mantissa for 404.

2. Since the characteristic of $\log N$ is -3, the first significant digit of N is in the 3d decimal place, by Rule II of Section 169: therefore, $N = .00404$.

Example 2. — Find N if $\log N = 3.6187$.

Solution. — In the table of mantissas, .6187 lies between the adjacent entries .6180 and .6191, which are the mantissas for 415 and 416, respectively. In the adjoining table we list the numbers whose logarithms are 3.6180 and 3.6191. $3.6191 - 3.6180 = .0011$ = tabular difference. $3.6187 - 3.6180 = .0007$ = partial difference.

(part. diff.) $\div$ (tab. diff.) $= \frac{7}{11}$.

Number	Logarithm
4150	3.6180
$N = ?$	3.6187
4160	3.6191

An increase of .0011 in the logarithm causes an increase of 10 in the number ($4160 - 4150 = 10$). Hence, by the principle of proportional parts, we assume that an increase of .0007 in the logarithm will produce an increase of $\frac{7}{11}(10)$ in the number. $\frac{7}{11}(10) = 6.4$, or 6, to the nearest unit. Hence, $N = 4150 + 6 = 4156$.

Note 1. — When determining a number N by use of a four-place table of logarithms, always state the result to *only* four significant digits because the fifth digit is generally ambiguous. Thus, in Example 2 we said $N = 4150 + 6 = 4156$, instead of saying $N = 4150 + 6.4 = 4156.4$.

Example 3. — Find N if $\log N = 7.3157 - 10$.

Solution. — 1. The mantissa 3157 is between the table entries 3139 and 3160.

$$\text{diff.} = 21. \left\{ \begin{array}{l} \text{diff.} = 18. \left\{ \begin{array}{l} 3139 \text{ is mantissa for } 2060 \\ 3157 \text{ is mantissa for } \quad ? \\ 3160 \text{ is mantissa for } 2070 \end{array} \right. \end{array} \right. \begin{array}{l} 2070 - 2060 = 10. \\ \frac{18}{21}(10) = 8.5^{+}, \text{ or} \\ \text{approximately } 9. \end{array}$$

Hence, 3157 is the mantissa for $2060 + 9 = 2069$.

2. Since the characteristic of $\log N$ is $(7 - 10)$, or -3, $N = .002069$.

Definition. *The number N whose logarithm is a given number L is called the* **antilogarithm** *of N.*

Thus, since $\log 41.5 = 1.6180$, the *antilogarithm* of 1.6180 is 41.5; since $\log 10{,}000 = 4$, the *antilogarithm* of 4 is 10,000.

EXERCISE 105

Find N if $\log N$ has the given value:

1. $\log N = 4.4183$.
2. $\log N = 3.6693$.
3. $\log N = 2.7324$.
4. $\log N = 5.9274$.
5. $\log N = 0.8331$.
6. $\log N = 0.8710$.
7. $\log N = 8.8797 - 10$.
8. $\log N = 7.1436 - 10$.
9. $\log N = 4.0815$.
10. $\log N = 5.2547$.
11. $\log N = 9.3350 - 10$.
12. $\log N = 6.6280 - 10$.

Find the antilogarithms of the following logarithms:

13. 1.2367.
14. $9.1395 - 10$.
15. 3.9276.
16. $9.6016 - 10$.
17. 0.5783.
18. $6.1349 - 10$.
19. 4.3899.
20. 2.3438.
21. $8.0738 - 10$.
22. 2.8862.
23. $7.1640 - 10$.
24. 0.8858.
25. $8.9935 - 20$.
26. 8.9429.
27. 3.1572.
28. 0.3169.
29. 2.4300.
30. $7.6715 - 10$.

175. Use of five-place logarithms;* supplementary.

Example 1. — Find $\log 25.637$.

Solution. — Notice that $25.630 < 25.637 < 25.640$.

$$\left. \begin{array}{l} \text{From the table: } \log 25.630 = 1.40875 \\ \text{From the table: } \log 25.640 = 1.40892 \end{array} \right\} \text{Tabular difference is } (.40892 - 40875) = .00017.$$

* The examples in the text will be solved with the aid of the five-place table in *Tables from the Mathematics of Investment*, by HART, published by D. C. HEATH AND COMPANY.

Since 25.637 is .7 of the way from 25.630 to 25.640, we assume that log 25.637 is .7 of the way from 1.40875 to 1.40892; .7 of the difference is .7(17) = .000119, which we change to .00012, the nearest number of five decimal places. Then, log 25.637 = 1.40875 + .00012 = 1.40887.

When determining a number N whose five-place logarithm is given, we agree **to state the result to only five significant digits**, because the sixth and later digits are ambiguous.

Example 2. — Find N if $\log N = 5.41152$.

Solution. — 1. *To determine the digits of N:*

diff. = 17 { diff. = 7 { 41145 is mantissa for 25790 ; 41152 is mantissa for ? ; 41162 is mantissa for 25800 } } | 25800 − 25790 = 10. ; $\frac{7}{17}(10) = 4$.

Hence, .41152 is the mantissa for 25790 + 4, or 25794.

2. The characteristic of log N is 5; hence, N = 257,940.

NOTE. — The column of **proportional parts**, headed **P.P.** in the table, should be used in interpolating, in order to avoid unnecessary arithmetic.

NOTE. — In using a five-place table, to avoid unnecessary arithmetic in interpolation, when N has more than five significant digits change N to the nearest number of five significant digits, before finding the logarithm of N. Thus, if N = 13.91372, we change it to N = 13.914.

SUPPLEMENTARY EXERCISE 106

Find the five-place logarithm of each number:

1. 198.7.	**5.** .13798.	**9.** .0035729.	**13.** 151.115.
2. .2633.	**6.** .018563.	**10.** .0047178.	**14.** .0111178.
3. 256.32.	**7.** .073563.	**11.** 13.1369.	**15.** 384.223.
4. 42.219.	**8.** 5321.7.	**12.** 5.62134.	**16.** .0000316.

Find the antilogarithms of the following five-place logarithms:

17. 4.52153.	**20.** 7.89003 − 10.	**23.** 0.35217.
18. 2.76938.	**21.** 3.21388.	**24.** 1.39875.
19. 8.87035 − 10.	**22.** 5.21631.	**25.** 8.65267 − 10.

26. 8.95321 − 10. **29.** 1.31352. **32.** 6.40998.

27. 4.32111 − 10. **30.** 0.86578. **33.** 2.13956 − 10.

28. 9.64397 − 10. **31.** 2.47120. **34.** 8.89532 − 20.

176. Computation of products and quotients.

Example 1. — Compute: (.0631)(7.208)(.51272).

Solution. — We use Property I of Section 166.

log .0631 = 8.8000 − 10
log 7.208 = 0.8578
log .5127 = 9.7099 − 10
(add) log of prod. = 19.3677 − 20 = 9.3677 − 10.

Product = .2332. (The number whose log is 9.3677 − 10.)

Example 2. — Compute $q = \dfrac{4.803 \times 269.9 \times 1.636}{7880 \times 253.6}$.

Solution. — From Property II of Section 166, log q equals the logarithm of the numerator minus the logarithm of the denominator. Both numerator and denominator are products whose logarithms are determined by Property I.

log **4.803** = 0.6815	log **7880** = 3.8965
log **269.9** = 2.4312	log **253.6** = 2.4041
log **1.636** = 0.2138	(add) log of den. = 6.3006
(add) log of num. = 3.3265	
log of denom. = 6.3006	
(subtract) log q = ??	

We recognize that, after performing the subtraction, log q will be negative. To obtain log q in the standard form for negative logarithms, we first add 10 to log q and then subtract 10 to compensate:

$$\begin{array}{rl} \log \text{ of numer.} = 3.3265 = & 13.3265 - 10 \\ \log \text{ of denom.} = & \quad 6.3006 \\ \hline \text{(subtract) } \log q = & \quad 7.0259 - 10 \end{array} \qquad (1)$$

Therefore, q = .001061 (the number whose logarithm is 7.0259 − 10).

Note 1. — Remember the device met in equation 1. *Whenever it is necessary to subtract a greater logarithm from a less, or a negative logarithm from a positive one, increase the characteristic of the minuend by* 10 *and then subtract* 10, *to compensate.*

NOTE 2. — The first operation in solving Example 2 was to write down the **computing form**, which consists of the part of the solution in heavy black type. Such a preliminary form is essential, in preventing errors. In illustrations, the computing form will be given in heavy black type.

EXERCISE 107

*Compute either by four-place or by five-place logarithms, as the instructor directs:**

1. 32.51×71.63.
2. $925.612 \times .000217$.
3. $84.75 \times .00368 \times .023517$.
4. $57 \times .789 \times 1.16$.
5. $\dfrac{728.72}{895}$.
6. $\dfrac{753}{.927}$.
7. $\dfrac{23.5276 \times 1.0513}{.143 \times .8972}$.
8. $\dfrac{16.083 \times 256}{47}$.
9. $\dfrac{1}{395.23 \times .079}$.
10. $\dfrac{16.8 \times 256.388 \times .00038}{264.25 \times 9.39}$.
11. $\dfrac{.29 \times .038 \times .0065}{1006.332 \times 2.71}$.
12. $\dfrac{5.6 \times 3.9078 \times .00031}{132 \times 1.93}$.
13. If $a = 39.526$, $b = 1321.7$, and $c = 58.3913$, compute

$$\frac{35\,ab}{bc - 3000}.$$

HINT. — Compute bc by use of logarithms. Then, insert the value of bc in the fraction, and compute the quotient.

14. Compute the reciprocal of 985; of 13.9218; of .0035628.

177. Computation of powers and roots. — We have Property III and equation 4 of Section 166 available for use.

Example 1. — Compute $(.3156)^4$.

Solution. — $\log (.3156)^4 = 4 \log (.3156) = 4(9.4991 - 10)$.
$\log (.3156)^4 = 37.9964 - 40 = 7.9964 - 10$.
Therefore, $(.3156)^4 = .009918$. (The antilog of $7.9964 - 10$.)

* If four-place logarithms are used, any number of five or more significant digits should be changed to the nearest number of four significant digits. If five-place logarithms are used, any number of six or more significant digits should be changed to the nearest number of five significant digits.

Example 2. — Compute $\sqrt[3]{856.3}$.

Solution. — $\log \sqrt[3]{856.3} = \dfrac{\log 856.3}{3} = \dfrac{2.9327}{3} = 0.9776.$

Therefore, $\sqrt[3]{856.3} = 9.498.$ (The antilogarithm of 0.9776.)

Example 3. — Compute (*a*) $\sqrt[6]{.08351}$; (*b*) $\sqrt[3]{.08351}$.

Solution. — (*a*) $\log \sqrt[6]{.08351} = \dfrac{\log .08351}{6} = \dfrac{8.9217 - 10}{6}.$

$\log \sqrt[6]{.08351} = 1.4870 - 1.6667 = -.1797$, which is inconvenient because it is not in the standard form for a negative logarithm. In order that the result after division by 6 may be in the standard form, we change $(8.9217 - 10)$ by subtracting 50 from (-10) to make it (-60), and by adding 50 to 8.9217, to compensate for the subtraction. Then

$$\log \sqrt[6]{.08351} = \frac{8.9217 - 10}{6} = \frac{58.9217 - 60}{6} = 9.8203 - 10.$$

Therefore, $\sqrt[6]{.08351} = .6612.$ (The antilogarithm of $9.8203 - 10$.)

Note 1. — Remember the device just used. *Before dividing a negative logarithm, write it in such a form that the negative part may be divided exactly by the divisor and give* -10 *as the quotient.*

Solution. — (*b*) $\log \sqrt[3]{.08351} = \dfrac{\log .08351}{3} = \dfrac{8.9217 - 10}{3}$

$$\log \sqrt[3]{.08351} = \frac{28.9217 - 30}{3} = 9.6406 - 10.$$

$$\sqrt[3]{.08351} = .4371.$$

EXERCISE 108

Compute by four-place or by five-place logarithms as the instructor directs:

1. $(175)^3$.	**5.** $\sqrt[5]{.0797}$.	**9.** $(.006382)^{\frac{1}{4}}$.	**13.** $(139)^{-2}$.
2. $\sqrt[3]{169}$.	**6.** $(56.731)^4$.	**10.** $\sqrt{531.263}$.	**14.** $(10.5178)^{-4}$.
3. $(45.63)^{\frac{2}{3}}$.	**7.** $(.013827)^2$.	**11.** $\sqrt[7]{.0039}$.	**15.** $(1.05)^{-4}$.
4. $(1.5672)^4$.	**8.** $\sqrt[4]{.389}$.	**12.** $\sqrt{16{,}300}$.	**16.** $\sqrt[6]{.0001}$.

17. $\left(\dfrac{56.3 \times 4.317}{21.4 \times \sqrt{521.923}}\right)^{\frac{3}{4}}$. **18.** $\dfrac{(152)^3 \times 25.73}{1893.32 \times 358}$.

HINT. — The computing form for Problem 17 is as follows:

$\log 56.3 =$
$\log 4.317 =$
(add) $\log$ of num. $=$
$\log$ of denom. $=$
(subt.) $\log$ of fract. $=$
$\frac{3}{4}$ ($\log$ of fract.) $=$

$\log 521.9 =$
(add) $\left\{\begin{array}{r}\frac{1}{2}\log 521.9 = \\ \log 21.4 = \end{array}\right.$
$\log$ of denom. $=$

Result $=$

19. $\dfrac{\sqrt[3]{.0198}}{(382.61)^2}$. **21.** $\sqrt{\dfrac{89.1}{163 \times .62}}$. **23.** $\dfrac{.031 \times 7.57938}{.221\sqrt{.387}}$.

20. $\dfrac{(54.1)(1.03)^5}{4756.3}$. **22.** $\sqrt{\dfrac{315 \times .198}{75.6392}}$ **24.** $\sqrt[3]{\dfrac{(531.7^2)(1.89)}{1573.82}}$.

25. $\sqrt[3]{\sqrt{.3985}}$. **26.** $(-5.893)^5(-2.57)^2$.

HINT. — In Problem 26, first compute as if all signs were positive, *as is necessary if logarithms are to be used.* Then, by inspection of the problem determine what sign should be given to the result.

27. $\dfrac{\sqrt[3]{-392.1}}{\sqrt[5]{-532}}$. **28.** $\dfrac{187.51(-1.62)^5}{389}$.

SUPPLEMENTARY PROBLEMS

29. $\dfrac{(35.6)^2 + 89.532}{\sqrt{11.1} - 2.513}$. **30.** $\dfrac{19.5 \times 5.78 - \sqrt[3]{.392}}{727}$.

HINT. — In Problem 29, first compute $(35.6)^2$ and $\sqrt{11.1}$ by use of logarithms; then insert their values in the fraction.

31. $\dfrac{\sqrt{8963} + .567}{532 - (115.3)^2}$. **33.** $\dfrac{(1.03)^5 - 1}{(1.03)^{\frac{1}{2}} - 1}$.

32. $\dfrac{\sqrt[3]{-29.3} + 21.5}{56 + \sqrt{.19583}}$. **34.** $\dfrac{1 - (1.04)^{-2}}{.0432}$.

35. (*a*) 85×63; (*b*) (log 85) $\times$ (log 63). **36.** $(3.57)^{1.6}$.

37. (*a*) $\dfrac{385}{532}$; (*b*) $\dfrac{\log 385}{\log 532}$. **38.** $\dfrac{\log 75 + 3}{\log 63.6}$.

39. $\dfrac{\log .88 - 2}{654}$. **40.** $\dfrac{893(\log .8)^2}{657}$.

41. $(71.38)^{.85}$. **43.** $(.85)^{-1.36}$. **45.** $(.328)^{2.13}$.

42. $(.7325)^{.78}$. **44.** $(.065)^{.532}$. **46.** $(.059)^{4.38}$.

HINT. — In Problem 42, $\log .7325 = 9.8648 - 10 = -.1352$.

$\log (.7325)^{.78} = .78(-.1352) = -.1055 = 9.8945 - 10$.

MISCELLANEOUS EXERCISE 109

Solve for x, computing square roots with the aid of logarithms:

1. $3.5x^2 - 1.327x - 2.1 = 0$. **2.** $8.6x^2 - .532x - .21 = 0$.

Definition. The **geometric mean** of n numbers is defined as the nth root of the product of the numbers. Thus, the geometric mean of M, N, P, Q, and R is $\sqrt[5]{MNPQR}$.

In each problem, find the geometric mean of the given numbers:

3. .00138; 19276; .08356; .0131.

4. 139; 395; 426; 537; 612.

5. 367; 157; 321; 598.

6. .85; 73; 62; 21.

7. If a, b, and c are the three sides of a triangle, it is proved in trigonometry that A, the area of the triangle, is given by $A = \sqrt{S(S-a)(S-b)(S-c)}$, where $S = \frac{1}{2}(a+b+c)$. Find the area of a triangle whose sides, in feet, are 375.4, 141.367, and 451.2.

8. The time t in seconds for one oscillation of a simple pendulum whose length is l centimeters, is given by the formula

$$t = \pi\sqrt{\frac{l}{980}}. \qquad (\pi = 3.1416)$$

Find the time t for a pendulum 165 centimeters long.

9. By using the formula of Problem 8, find the length of a pendulum whose time of oscillation is 2 seconds.

10. How many cubic feet are there in a room whose dimensions in feet are 13.75 by 62.2 by 12.3?

11. The volume of a right circular cylinder is $V = \pi r^2 h$ where r is the radius of the base and h the height. How many liquid gallons are contained in a cylinder whose height is 160 feet and whose base has a radius of 4.85 feet?

HINT. — Express the final result as a fraction before using logarithms. Recall that one gallon contains 231 cubic inches.

12. It has been determined that, if a white oak beam, supported at both ends, is uniformly loaded, the maximum safe load W in pounds is

$$W = \frac{4000\ BD^2}{3\ L},$$

where L is the distance between the supports, B the breadth of the beam, and D its depth. L, B, and D are measured in inches. Find the safe load for a beam 5 inches wide and 4 inches deep if the supports are 159 inches apart.

13. A white oak beam 6 inches wide is supported at both ends, and is to be uniformly loaded. If the distance between supports is 133 inches, and if the maximum safe load desired is 3000 pounds, find the proper depth for the beam, by use of the formula of Problem 12.

14. If a body has fallen s feet from rest in a vacuum, the velocity v of the body in feet per second is given by $V^2 = 2\ gs$, where $g = 32.16$. Find the velocity of a body which has fallen 1740 feet.

SUPPLEMENTARY TOPICS

178. Exponential equations. — An equation such as $3^x = 7$, where the unknown occurs in an exponent, is called an **exponential equation.** In solving an exponential equation, try to put it in such a form that, after the logarithms of the two members are equated, an ordinary algebraic equation is obtained.

Example 1. — Solve $13^{2x+2} = (356)5^x$.

Solution. — 1. Equate the logarithms of the two sides:

2. $(2\ x + 2) \log 13 = \log 356 + x \log 5.$

3. $(2x + 2)(1.1139) = 2.5514 + .6990x.$

4. $2.2278x + 2.2278 = 2.5514 + .6990x.$

5. $1.5288x = .3236.$

6. $x = \dfrac{.3236}{1.5288} = .2116.$

$\log .3236 = 9.5100 - 10$
$\log 1.529 = 0.1844$
(subtract) $\log x = 9.3256 - 10$

EXERCISE 110

Solve the following equations:

1. $12^x = 38.$
2. $51^x = 569.$
3. $5^x = 27(2^x).$
4. $15^x = 95(3^x).$
5. $(1.05)^n = 6.3251.$
6. $6^{(x^2-2x)} = 18.$
7. $\dfrac{(1.05)^n - 1}{.05} = 6.3.$
8. $\dfrac{1 - (1.03)^{-n}}{.03} = 2.4.$
9. Solve for x and y: $\begin{cases} 3^x 5^y = 15. \\ 2^x = 12(4^{2y}). \end{cases}$

179. Graphs of logarithmic and exponential functions. The graph of $y = \log_a x$ is also the graph of $x = a^y$, because $y = \log_a x$ means that $x = a^y$.

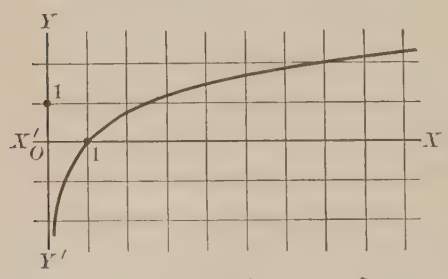


FIG. 27.

Illustration. Figure 27 is a graph of $y = \log_a x$ when $a = 2.71828\cdots$, but the general characteristics of the graph would not be different if a had any other value, > 1. In graphing $y = \log_a x$, a table of Naperian logarithms was used in finding pairs of values of x and y. The graph in Figure 27 assists one in remembering the following facts:

I. *If x is negative, $\log_a x$ is not a real number.*

II. *If $0 < x < 1$, $\log_a x$ is negative.*

III. *If a has any value, $\log_a 1 = 0$.*

IV. *If x increases without limit, $\log_a x$ increases without limit; if x approaches zero, $\log_a x$ decreases without limit.*

EXERCISE 111

1. Graph the function $y = \log_{10} x$ for $0 < x \leqq 30$.

2. Graph $y = 10^x$ between $x = -4$ and $x = 1$. From the graph, read off $\log 5$; $\log 7$; $\log \frac{1}{2}$; $\log .2$.

3. (*a*) Graph $y = 2^x$ from $x = -5$ to $x = 5$. (*b*) From the graph, read off $\log_2 5$; $\log_2 10$; $\log_2 .5$.

4. Graph $y = 2^{x^2}$ from $x = -5$ to $x = 5$.

180. Logarithms to bases different from 10. — If $x = \log_a N$, then $a^x = N$. Hence, if N and a are given, we can find $\log_a N$ by solving the exponential equation $a^x = N$ for x, by use of common logarithms.

NOTE. — Recall that the base of the *natural system* of logarithms is $e = 2.71828$. The following logarithms will be useful:

$$\log_{10} e = 0.43429. \qquad \log_{10} .43429 = 9.63778 - 10. \tag{1}$$

Example 1. — Find $\log_e 35$.

Solution. — 1. Let $x = \log_e 35$. Then, $e^x = 35$.

2. Take the common log of both sides: $x \log_{10} e = \log_{10} 35$.

3. $x = \dfrac{\log_{10} 35}{\log_{10} e} = \dfrac{1.5441}{0.4343}$

$\log 1.544 = 10.1886 - 10$

$\log .4343 = 9.6378 - 10$

(subtract) $\log x = 0.5508$

$x = 3.555 = \log_e 35$.

EXERCISE 112

1. Find $\log_e 75$; $\log_e 10$; $\log_6 78$; $\log_{15} 33$; $\log_{12} 1000$.

2. Find the natural logarithm of 4368; of .85.

3. Solve for x: $\log_3 25 = 2x$.

4. Find the logarithm of .001 to the base e.

Theorem. *If a and b are any two bases, then*

$$\log_a N = (\log_a b)(\log_b N). \tag{2}$$

Proof. — **1.** Let $y = \log_b N$; then $N = b^y$.

2. Hence, $\log_a N = \log_a b^y = y \log_a b = (\log_a b)(\log_b N)$.

The number $\log_a b$ is called the **modulus** of the system of base a with respect to the system of base b. Given a table of logarithms to the base b, we could form a table of logarithms to the base a by multiplying each entry of the given table by the modulus, $\log_a b$.

5. Find the modulus of the common system with respect to the natural system of logarithms.

6. Prove that $\log_a b = 1 \div \log_b a$, by use of equation 2.

In problems 7 *to* 9, *find the modulus described:*

7. Of the natural system with respect to the Briggs system.

8. Of the Briggs system with respect to the system to the base 5.

9. Of the system to the base .5 with respect to the system of common logarithms.

10. (*a*) By use of the result of Problem 9, find $\log_{.5} 100$; $\log_{.5} 10{,}000$; $\log_{.5} .1$.

11. By use of the result of Problem 9, draw a graph of $y = \log_{.5} x$.

CHAPTER XXII

PERMUTATIONS AND COMBINATIONS

181. Fundamental principle. — If one thing can be done in h different ways and if after it is done in any of these ways, a second thing can be done in any one of k different ways, then the two things can be done in the stated order in hk different ways.

Proof. — For each way of doing the first thing there are k different ways of doing the two things in the stated order. Hence, since there are h ways of doing the first thing, there are hk ways of doing the two things in the stated order.

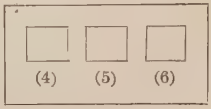
Example 1. — Suppose there are four railroads connecting A and B and five connecting B and C. In how many ways can we select a route for traveling from A to B to C?

Solution. — Corresponding to each of the 4 ways of going from A to B there are 5 ways to continue from B to C. Hence, there are $5 \cdot 4$, or 20, different routes from A to B to C.

An evident extension of Theorem I holds if three or more different acts are involved. Thus, *if one thing can be done in h ways, and then a second thing in k ways, and then a third thing in l ways, the three things can be done together in the stated order in hkl different ways.*

Example 2. — How many numbers * of three different digits each can be formed from the digits 1, 2, 3, 5, 8, and 9?

Solution. — We can choose any one of the six digits for the units' place, then, any one of the five remaining digits for the tens' place, and then, any one of the four remaining digits for the hundreds' place. Hence, we can form $6 \cdot 5 \cdot 4$ (or, 120) different numbers of the specified kind.



* In this chapter, by a *number* we shall mean a *positive integral number.*

Example 3. — In Example 2, how many of the numbers formed will be odd?

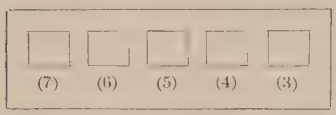
Solution. — In forming an odd number, the units' place can be filled in any one of four ways (by 1, or 3, or 5, or 9). Then, we can select any one of the five remaining digits for the tens' place, and then any one of the four remaining digits for the hundreds' place. Hence, we can form $4 \cdot 5 \cdot 4$ (or, 80) different odd numbers.

182. Permutations. — We can think of any or all of a set of things as being arranged in a set of numbered places; for instance, in a line where the places are numbered 1, 2, 3, ··· from left to right. Any such ordered arrangement of all or of any part of a set of things is called a **permutation** of them. If r of the things occur in the arrangement, it is called a permutation of the things *taken r at a time*.

Thus, the different permutations of the letters *a*, *b*, and *c*, taken two at a time, are *ab*, *ba*, *ac*, *ca*, *bc*, and *cb*; their permutations, taken three at a time, are *abc*, *acb*, *bac*, *bca*, *cab*, and *cba*.

Example 1. — Find the number of permutations of seven different books, taken five at a time.

Solution. — In any permutation, we can fill the first place by any of the seven books; then, the second place by any of the six books remaining after the first place is filled; then, the third place by any of the five remaining books; then, the fourth place by any of the four remaining books; then, the fifth place by any of the three remaining books. Hence, by Section 181, we can form permutations of the books, taken five at a time, in $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$ (or, 2520) ways.



EXERCISE 113

1. How many numbers of four different digits each can be formed from the digits 1, 2, 5, 6, 7, and 8?

2. In how many ways can the offices of president, vice-president, and secretary be filled by selections from 15 people?

3. How many permutations are there of the letters in the word *hotel* (*a*) taken five at a time; (*b*) taken four at a time?

4. In how many ways can a teacher seat six pupils if there are eight seats available?

5. How many odd numbers of four different digits each can be formed from the digits 2, 3, 4, 5, 7, and 9?

6. An employment agency has four applicants with whom to fill three different positions. In how many ways can the positions be filled?

7. (*a*) Find the number of permutations of four letters x, y, z, and w, taken 4 at a time. (*b*) Write out all of these permutations.

8. How many numbers, divisible by 5, of three different digits each, can be formed from the digits 1, 3, 5, 4, 6, and 7?

9. In how many ways can a teacher seat three girls and three boys in a row of six seats if the boys are to have the 1st, 3d, and 5th seats?

10. From the digits 2, 4, 6, 3, 5, and 7, how many numbers of six different digits each can be formed in which the even and odd digits alternate?

11. If 5 coins are tossed together, in how many different ways can they fall?

12. How many words of five different letters each can be formed from the letters of the word *after?*

NOTE for Problem 12. — Here and elsewhere, assume that each permutation of a set of letters forms a word, although this is not strictly true.

13. In how many different ways can a man distribute a nickel and a dime among six boys?

14. In how many ways can two men take up quarters in a city where there are five hotels?

15. A girl has invited six friends to a dinner party. After locating herself at the table, in how many ways can she arrange her guests in the remaining six seats?

16. Two cubical dice are tossed together. In how many different ways can they fall?

17. The coach of a baseball team of 9 men has only one man who can catch, and only two others who can pitch. If each of the other men and the pitchers can play any of the other positions, in how many ways can the coach assign his men to the nine positions on the team?

18. Find the number of permutations (*a*) of 10 letters taken three at a time; (*b*) of n letters taken four at a time.

19. How many numbers of five different digits each can be formed from the digits 0, 2, 3, 4, 5, and 7?

HINT. — The zero is not eligible for the left-most place. Fill the places from left to right.

20. How many numbers can be formed by use of the digits 1, 2, 3, and 5, if no digit is used more than once in a number?

21. How many different numbers can be formed by use of the digits 2, 3, 5, 7, and 8, if no digit is used more than once in a number?

22. How often can four people take seats on a bench without sitting twice in the same order?

23. In how many ways can a person place twelve different books on a shelf if three particular books are to be placed side by side?

HINT. — Places *in which to arrange* these three books can be selected in 10 ways. (Why?)

24. How many numbers greater than 5000 of four different digits each can be formed from the digits 0, 3, 6, 7, 4, and 2?

25. An autobus has five seats on the left side and seven on the right side. In how many ways can seven persons seat themselves, if only a certain four decide to sit on the right side?

26. From the letters of the word *volume*, how many words of six different letters each can be formed, in which vowels and consonants alternate, with a vowel at the left end?

183. Formulas. — Let ${}_nP_r$ represent the number of permutations of n different things taken r at a time.

Thus, "${}_5P_4$" is read "the number of permutations of 5 different things taken 4 at a time."

Theorem I. — *The number of permutations of n different things taken r at a time is* $n(n-1)(n-2)\cdots(n-r+1)$;

or,
$$\boldsymbol{{}_nP_r = n(n-1)(n-2)\cdots(n-r+1).} \qquad (1)$$

Proof. — In any permutation, we can fill the first place by any of the n different things, then, the second place by any of the $(n-1)$ things remaining after the first place is filled, then, the third place by any of the $(n-2)$ things remaining, $\cdots$, finally, the rth place by any of the $(n-r+1)$ things remaining after the $(r-1)$th place is filled. Hence, by Section 181, all r places in a permutation of the n things, taken r at a time, can be filled in $n(n-1)(n-2)\cdots(n-r+1)$ different ways.

Thus, ${}_7P_4 = 7\cdot 6\cdot 5\cdot 4 = 840$; ${}_nP_3 = n(n-1)(n-2)$; etc.

When we place $r = n$ in (1) we obtain

$$\boldsymbol{{}_nP_n = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!.} \qquad (2)$$

That is, *the number of permutations of n different things taken n at a time is n factorial.*

184. Permutations of things not all different.

Example 1. — Find the number of permutations of the letters a, a, b, b, b, taken 5 at a time.

Solution. — 1. Let P be the number of permutations.

2. Replace the two a's by a_1, a_2, and the three b's by b_1, b_2, b_3, where these new letters are all different.

3. Consider any one of the P permutations of a, a, b, b, b, taken 5 at a time. In the two places occupied by the a's we can arrange a_1 and a_2 in 2! different ways, and in the three places occupied by the b's, we can permute b_1, b_2, and b_3 in 3! different ways. Hence, by Section 181, from each of the P permutations of a, a, b, b, b, we can obtain $2!\cdot 3!$ different

permutations of a_1, a_2, b_1, b_2, b_3. Hence, the total number of permutations of these letters is $P \cdot 2! \cdot 3!$. But, the number of permutations of five different letters taken 5 at a time is $5!$. Hence,

$$P \cdot 2! \cdot 3! = 5!, \text{ or } P = \frac{5!}{2! \cdot 3!} = 10.$$

By the method of the preceding solution, we can prove

Theorem I. — *If P represents the number of distinct permutations of n things, taken all at a time, when, of the n things, there are n_1 alike, n_2 others alike, n_3 others alike, etc., then*

$$P = \frac{n!}{n_1!\ n_2!\ n_3! \cdots}. \tag{1}$$

Proof. — **1.** For simplicity, suppose that the n things consist of only two kinds of things, n_1 of one kind and n_2 of the other. Let the n things be represented by letters, consisting of n_1 a's and n_2 b's.

2. Consider n new letters, all different, obtained by replacing the n_1 a's by $a_1, a_2, \cdots, a_{n_1}$ and the n_2 b's by $b_1, b_2, \cdots, b_{n_2}$. Then, from each of the P permutations of the original letters, we can obtain $(n_1!)(n_2!)$ different permutations of the new letters, by permuting $a_1, a_2, \cdots, a_{n_1}$ in the places previously occupied by the n_1 a's, and $b_1, b_2, \cdots, b_{n_2}$ in the places previously occupied by the n_2 b's. Hence, the total number of permutations of the new letters, taken n at a time, is $P \cdot n_1! \cdot n_2!$ But, the number of permutations of these n different letters, taken n at a time, is $n!$. Hence,

$$P \cdot n_1! \cdot n_2! = n!, \text{ or } P = \frac{n!}{n_1!\ n_2!}. \tag{2}$$

The general case of formula 1 can be established just as we have established (2).

EXERCISE 114

In Problems 1 *to* 4 *find the number of distinct permutations:*

1. Of twelve things taken four at a time.
2. Of six things taken six at a time.
3. Of the letters a, a, a, b, b, b, c, c, taken eight at a time.
4. Of the digits 3, 3, 4, 5, 6, taken five at a time.

5. In how many different orders can the colors red, blue, green, and yellow be arranged taken four at a time?

6. How many distinct permutations can be made of the letters of the word *attention* taken all at a time?

7. In how many different relative orders can seven people take seats at a round table?

HINT. — Think of the position of any one of the seven as being fixed.

8. How many numbers of seven digits each can be formed by use of the digits 2, 2, 3, 3, 5, 4, and 4?

9. In how many distinct ways can four dimes and six quarters be distributed among ten children if each child is to receive a coin?

10. If we have four white billiard balls, exactly alike, and three red balls, exactly alike, in how many distinct ways can all of the balls be arranged in a line?

11. In making a signal with colored flags, the flags are arranged in order from left to right. How many different signals can be made with seven flags if all are raised at once, if two flags are red, three are white, and two are blue?

185. A combination of a set of things is a group of all or of any part of the things, **without regard to the order of the things in this group.** A combination involving r of the things is called a combination of the things *taken r at a time.*

Illustration 1. Thus, the different combinations of a, b, c, and d, taken 3 at a time, are (a, b, c), (a, b, d), (a, c, d), and (b, c, d). From each combination, we can form 3! or 6 different permutations of the 4 letters taken 3 at a time. Thus, from (a, b, c) we can form the permutations *abc, acb, bac, bca, cab,* and *cba.* In other words, there are only *four combinations* of 4 different letters taken 3 at a time, whereas there are $4 \cdot 6$, or 24, *permutations* of the 4 letters taken 3 at a time.

We use the symbol ${}_nC_r$ to denote the number of combinations of n different things taken r at a time.

Thus, ${}_7C_3$ represents the number of combinations of 7 different things taken 3 at a time.

Theorem I. — *The number of combinations of n different things taken r at a time equals the number of permutations of n different things, taken r at a time, divided by $r!$ That is,*

$$ {}_nC_r = \frac{{}_nP_r}{r!}. \tag{1}$$

Proof. — With each combination, containing r of the things, we can form $r!$ permutations of the things taken r at a time. Hence, since there are ${}_nC_r$ different combinations, there are ${}_nC_r \cdot (r!)$ different permutations. That is, ${}_nP_r = {}_nC_r \cdot (r!)$, or

$$ {}_nC_r = \frac{{}_nP_r}{r!}. $$

Note 1. — Since ${}_nP_r = n(n-1)(n-2)\cdots(n-r+1)$, hence,

$$ {}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}. \tag{2}$$

If both numerator and denominator in (2) are multiplied by $(n-r)!$, the new numerator is $1 \cdot 2 \cdot 3 \cdots (n-r)(n-r+1)\cdots(n-2)(n-1)n$, or, $n!$, and hence

$$ {}_nC_r = \frac{n!}{r!\,(n-r)!}. \tag{3}$$

Note 2. — Recognize that expression 2 for ${}_nC_r$ is the same as the coefficient of $x^{n-r}y^r$ in formula 4, page 181. Hence,

$$ (x+y)^n = x^n + {}_nC_1\, x^{n-1}y + {}_nC_2 x^{n-2}y^2 + \cdots + {}_nC_r x^{n-r}y^r + \cdots + {}_nC_n y^n. \tag{4}$$

In (4), if we place $x = 1$ and $y = 1$, we obtain

$$ 2^n = 1 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_r + \cdots + {}_nC_n, $$

or,
$$ {}_nC_1 + {}_nC_2 + \cdots + {}_nC_n = 2^n - 1. \tag{5}$$

Thus, *the total number of combinations of n things taken 1 at a time, or 2 at a time, $\cdots$, or n at a time, is* $(2^n - 1)$.

Note 3. — Whenever we pick r things from n things, we leave, or set aside, $(n-r)$ of the things. Hence, the number of combinations

of n different things taken r at a time is the same as the number of combinations of n different things taken $(n - r)$ at a time. That is,

$$_nC_r = {}_nC_{n-r}. \tag{6}$$

Formula 6 can also be proved by use of (3). From (3),

$$_nC_{n-r} = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!\,r!} = {}_nC_r.$$

MISCELLANEOUS EXERCISE 115

1. By the method of Illustration 1, page 271, find the number of combinations of five letters a, b, c, d, and e, taken four at a time.

2. (a) Write out all of the combinations of a, b, c, d, and e, taken 4 at a time. (b) Write at least 12 different permutations of a, b, c, d, and e, taken 4 at a time.

3. How many different groups of 5 people each can be selected from 11 people?

4. How many different sums of money, consisting of three coins each, can be formed from a dime, a dollar, a quarter, and a penny?

5. How many different hands of 13 cards each can be made from a deck of 52 cards? Leave the result in factored form.

6. From a group of 10 friends, in how many different ways can a man choose a dinner party of 8 friends?

7. From a group of 6 different things, in how many ways can we choose groups (a) of exactly 2 things each; (b) of at least 2 things each?

HINT for (b). — Find the number of groups of 2, of 3, of 4, of 5, etc.

8. How many different sums of at least three coins each can be formed from a penny, a nickel, a dime, and a quarter?

9. (a) How many different straight lines can be drawn through points selected from 12 points in a plane, if no three of the points are in a line? (b) How many of these lines will go through any one point?

10. (*a*) In how many ways can a committee of 5 men be selected from a group of 10 men? (*b*) On how many of these committees will any particular man be found?

11. In how many ways can we select groups of 8 books consisting of 5 English books and 3 German books, from 10 different English books and 6 different German books?

HINT. — In how many ways can you select 5 English books from 10 of them? In how many ways can you select 3 German books from 6 of them? Use Section 181.

12. There are 6 roads between A and B. In how many different ways can a person make the round trip from A to B and back to A?

13. A bag contains 6 black and 5 white balls. In how many ways can we draw from the bag (*a*) a group of 4 black balls; (*b*) a group of 4 balls; (*c*) a group of 4 balls of which 2 are white and 2 are black?

14. Out of 5 sailors and 10 soldiers, how many different groups can be formed each consisting of 3 soldiers and 2 sailors?

15. In a baseball league of 8 teams, how many games will be played if each team plays 10 games with each other team?

16. How many different groups of 1500 people each can be formed from 1502 persons?

HINT. — Read Note 3, page 272.

17. How many words of 8 different letters each can be formed from 8 different letters if a certain two letters are to be consecutive in the words?

18. How many different triangles can be drawn by selecting their vertices from 11 points in a plane, if no 3 of the points are in a line?

19. In how many ways can a captain of a baseball team arrange his batting order, if he wishes a to fill places 1, 4, 7, and 8 by selections from a certain four of the men?

20. Four coins are tossed together. In how many ways can they fall?

21. Three dice are tossed together. In how many ways can they fall?

22. From a pack of 52 cards, two cards are drawn together. In how many ways can it result that one card is a king and one is a queen?

SUPPLEMENTARY PROBLEMS

23. A committee of five is to be selected from a group of 4 men and 5 women. In how many ways can the committee be chosen if it is to contain (*a*) exactly 3 men; (*b*) at least 3 men?

24. From the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9, how many numbers of five different digits each can be formed, if each number is to contain three odd and two even digits?

Solution. — 1. From the odd digits (1, 3, 5, 7, 9) we can select groups of three in ${}_5C_3$, or 10, ways. From the even digits (2, 4, 6, 8) we can select two in ${}_4C_2$, or 6, ways. Hence, by Section 181, we can form $10 \cdot 6$, or 60, *different groups*, each containing three odd and two even digits.

2. *From each of these* 60 *groups*, we can form $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, or 120, different numbers of 5 digits each. Hence, we can form $60 \cdot 120$, or 7200, numbers of the specified sort.

25. Out of a group of 8 consonants and 6 vowels, how many words of 7 different letters each can be formed, each containing 3 consonants and 4 vowels?

26. From the digits 1, 2, 3, 4, 5, and 6, how many numbers of four different digits each can be formed, each consisting of two odd and two even digits?

27. In how many ways can 15 different books be divided so that A, B, and C shall receive 6, 5, and 4, respectively?

28. By use of formula 4, page 272, write the expansion of $(x + y)^8$.

29. How many numbers of six different digits each can be formed from six different digits, if a certain two of the digits are not to be consecutive in any number?

30. If ${}_nP_4 = 840$, find the value of ${}_nC_4$.

31. How many odd numbers of five different digits each can be formed by use of the digits 0, 1, 3, 5, 4, and 6?

CHAPTER XXIII

PROBABILITY

186. Probability. — By a **trial** of an event we shall mean an act which gives the event an opportunity to happen or to fail (to happen). In the following definition we consider an event which, at each trial, can happen in h ways and fail in f ways, where *all ways are equally likely.*

Definition 1. — *At any trial of an event, the probability p of it happening is the ratio of the number of ways of happening to the total number of ways of happening or failing, or*

$$p = \frac{h}{h+f}, \tag{1}$$

and the probability q of the event failing is the ratio of the number of ways of failing to the total number of ways of happening or failing, or

$$q = \frac{f}{h+f}. \tag{2}$$

Thus, if a bag contains 7 black and 3 white balls, the probability that a ball, drawn at random, will be black is $\frac{7}{10}$, because a black can be drawn in 7 ways ($h = 7$) out of 10 ways of drawing a black or a white.

It is said that the **odds** are h to f *in favor* of the event if $h > f$, and h to f *against* the event if $h < f$.

From (1) and (2), we verify that $p + q = 1$; or, *the sum of the probabilities of happening and of failing is* 1.

NOTE. — In both (1) and (2), the numerator cannot exceed the denominator. Hence, *any probability is a positive number, not greater*

than 1. Moreover, if $f = 0$, then $p = \frac{h}{h} = 1$; hence, *when an event is certain to happen, its probability of happening is* 1. Similarly, *if an event is certain to fail, its probability of failing is* 1 *and its probability of happening is zero.*

Example 1. — From a bag containing 5 red and 6 white balls, 4 balls are drawn at random. Find the probability of drawing 2 reds and 2 whites.

Solution. — 1. *The favorable cases.* — The number of ways of drawing 2 reds from 5 reds is ${}_5C_2$, or 10, and of drawing 2 whites from 6 whites is ${}_6C_2$, or 15. Hence, the number of ways of drawing together 2 reds and 2 whites is $10 \cdot 15$, or 150.

2. *The total possible cases.* — The number of ways of drawing 4 balls from 11 balls is ${}_{11}C_4$, or 330.

3. The probability of drawing 2 reds and 2 whites is $\frac{150}{330}$, or $\frac{5}{11}$.

187. Probability determined by experiment. — If an event has been observed to happen h times out of t trials, where t is a large number, then, until further knowledge of the event is obtained, we may take $p = \frac{h}{t}$ as our estimate of the probability that the event will happen at a future trial.

NOTE. — This experimental method of determining a probability is very important in problems in statistics and insurance. In most cases, the estimate $p = \frac{h}{t}$ becomes increasingly reliable and approaches a limit as t, the number of observed cases, becomes infinite.

Example 1. — If a man is alive at the age of 25, find the probability (1) that he will live at least 13 years; (2) that he will die in his 43d year.

Solution. — 1. In this, and in similar examples, by a *man* we mean a *man selected at random*. In the American Experience Table of Mortality (page 350) we observe 89,032 men alive at age 25. Of these, 79,611 remain alive at age 38. Hence, the probability of the man aged 25 living at least 13 years is $\frac{79611}{89032}$.

2. Of the 89,032 alive at age 25, 785 die in their 43d year (see the table; 76,567 − 75,782 = 785). The probability of dying in the 43d year is $\frac{785}{89032}$.

188. Mathematical expectation. — If p is the probability of a person receiving a sum $\$S$, we call pS his **mathematical expectation.**

EXERCISE 116

1. A bag contains 6 white, 5 red, and 20 black balls. What is the probability that, if a ball is drawn,* it will be (*a*) black; (*b*) black or white?

2. From a pack of 52 cards, we draw a card. What is the probability that it will be (*a*) an ace; (*b*) an ace or a king?

3. If a coin is tossed, what is the probability that it will fall head up?

4. The faces of a cubical die are numbered 1, 2, 3, 4, 5, and 6, respectively. If a die is thrown, what is the probability (*a*) that 3 will turn up; (*b*) that 2 or 5 will turn up?

5. What is the mathematical expectation of a player in a game where his probability of winning is $\frac{3}{5}$ and the stake is \$100?

NOTE. — We may think of any probability as the probability of winning a game. Then, it is important to realize that when we say, for instance, "*the probability of winning a game is* $\frac{3}{5}$," we mean that (*a*) if a very large number of games are played, we can expect approximately $\frac{3}{5}$ of them to be won, and (*b*) if the number of games played becomes larger and larger without limit, the ratio of the number won to the number played will approach $\frac{3}{5}$ as a limit. We do *not* imply, for instance, that if 15 games are played, *exactly* $\frac{3}{5}$ of 15, or 9, will be won. If only a few games are played, *more, or equally well less*, than $\frac{3}{5}$ of the total may be won.

6. As a coöperative class exercise, make 1000 tosses of a coin and keep a record of how many fall heads out of the first (*a*) 10 trials; (*b*) 250 trials; (*c*) 1000 trials. In each case compare the number of heads with $\frac{1}{2}$ of the number of trials.

7. If the odds are 5 : 3 in favor of winning \$50, find the player's expectation.

* Here and elsewhere, *to draw* means to draw *at random*.

8. If the probability of winning a game is $\frac{1}{4}$, what is the probability of losing?

9. A bag contains 50 envelopes of which 20 contain \$5 each, while the others are empty. If we draw an envelope, find (*a*) the probability of drawing \$5; (*b*) our mathematical expectation.

10. Out of 1,000,000 buildings of a certain type, the equivalent of 2500 total losses will be suffered through fire inside of one year. An owner insures one of these buildings for \$20,000. Find the value of his expectation.

Find the probability in problems 11 *to* 14, *by use of the mortality table* (page 350):

11. That a boy aged 15 will be alive 10 years later.

12. That a man aged 33 will live at least 25 years.

13. That a man aged 56 will die (*a*) within 5 years; (*b*) during the fifth year.

14. That a man aged 35 will die within 1 year.

15. A bag contains 6 white balls, 4 red balls, and 5 black balls. Find the probability that

(*a*) if 1 ball is drawn, it will be white;
(*b*) if 1 ball is drawn, it will be either red or white;
(*c*) if 2 balls are drawn, both will be white;
(*d*) if 5 balls are drawn, there will be 2 red and 3 black;
(*e*) if 2 balls are drawn, both will be of the same color.

16. A bag contains 5 blue, 4 white, and 5 yellow tags. Find the probability that

(*a*) if 1 tag is drawn, it will be white or yellow;
(*b*) if 2 tags are drawn, both will be yellow;
(*c*) if 5 tags are drawn, 2 will be blue and 3 yellow;
(*d*) if 3 tags are drawn, all will be of the same color.

17. From a group of 15 men and 10 women, a committee of four is chosen by lot. Find the probability that the committee will consist (*a*) of 4 women; (*b*) of 2 men and 2 women.

18. Three letters are selected at random from *a*, *c*, *i*, *o*, *n*, and *m*. Find the probability that exactly two of the three are vowels.

Illustrative Example. — Six coins are tossed. Find the probability that exactly two will fall heads.

Solution. — 1. The total number of ways in which six coins can fall together is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, or 64.

2. *The favorable cases.* — Two coins, which are to fall heads, can be selected from six coins in ${}_6C_2$, or 15, ways. For each of these ways, there is only 1 way in which all of the other 4 coins can fall tails. Hence, there are $15 \cdot 1$, or 15, ways in which 2 heads and 4 tails can result. Hence, the desired probability is $\frac{15}{64}$.

19. Five coins are tossed. Find the probability that they will fall (*a*) all heads; (*b*) all tails; (*c*) 3 heads and 2 tails.

20. From a pack of 52 cards, we draw 3 cards. Find the probability that they consist (*a*) of an ace, a king, and a queen; (*b*) of an ace, a king, and a queen, all of the same suit.

21. The tickets in a box are numbered from 1 to 40, inclusive. If two tickets are drawn, find the probability that the sum of their numbers is even.

22. In a single throw with 4 coins, find the probability that they will fall with at least 3 heads.

HINT. — In how many ways can we obtain exactly 3 heads; exactly 4 heads?

23. In a single throw with two dice, find the probability of turning up a total (*a*) of 7; (*b*) of 10; (*c*) of at most 4.

HINT for (*a*). — A 7 can result from 1 on either die and 6 on the other; or, 2 on either and 5 on the other; etc.

24. From a pack of 52 cards, 3 cards are drawn. Find the probability that all are of the same suit.

25. On tossing 3 coins, what is the probability of getting (*a*) exactly 1 head; (*b*) at most 2 heads?

26. Find the probability of getting at least 1 head, if 4 coins are tossed.

HINT. — To find the number of favorable cases, first find the total number of possible cases, and then subtract the number of unfavorable cases. When there is *not* at least 1 head, there are 4 tails.

27. A first bag contains 10 white and 5 black balls and a second bag contains 4 white and 16 black balls. If one ball is drawn from each bag, find the probability (*a*) that both are white; (*b*) that one is white and one is black.

28. If we draw 2 balls from the first bag in Problem 27, and 1 ball from the second bag, what is the probability that all balls drawn are black.

29. A bag contains 10 black and 3 white balls. If we draw 2 balls in succession, find the probability that both are white, (*a*) if the first drawn is replaced before the second is drawn; (*b*) if the first ball is not replaced before the second is drawn.

30. If three balls are drawn in succession, from the bag of Problem 29, what is the probability that all are black, if the first ball drawn is replaced, but the second drawn is not replaced?

31. A bag contains twenty \$1 bills, ten \$5 bills, and fifteen \$10 bills. What is the expectation of a person who draws one bill?

HINT. — Find his expectation, separately, on account of the presence of each type of bills.

32. From the digits 2, 3, 5, 7, 8 we form a five-digit number at random. Find the probability that this number is divisible by 2.

189. Mutually exclusive events. — Consider a set of events, any of which may happen if a certain act is performed. The events of the set will be called **mutually exclusive** if the happening of any one excludes the happening of any other.

Theorem I. — *The probability that one or other of a set of mutually exclusive events will happen is the sum of the probabilities of happening for the separate events.*

Proof. — **1.** For simplicity, consider a set of only two events W and B, and suppose that any trial can result in u different ways, out of which W happens in m ways, and B in n ways.

For concreteness, think of any trial as the drawing of a ball from a bag containing u balls of which m are white and n are black, and consider W and B as the events of drawing a white and a black, respectively.

2. The probability of W happening is $\frac{m}{u}$ and of B happening is $\frac{n}{u}$.

3. The m ways in which W happens and the n ways in which B happens are all different, because W and B are mutually exclusive. Thus, *one or other* of W and B can happen in $(m + n)$ ways. Hence, the probability of one or other happening is $\frac{m+n}{u}$, or $\left(\frac{m}{u} + \frac{n}{u}\right)$, which we see is the *sum of the probabilities of happening of the separate events.*

Illustration. A bag contains 3 white, 7 black, and 10 yellow balls. If a ball is drawn, the probability that it is white is $\frac{3}{20}$ and that it is black is $\frac{7}{20}$. Hence, by Theorem I, if a ball is drawn, the probability that it is *either* white *or* black is $(\frac{3}{20} + \frac{7}{20})$, or $\frac{10}{20}$, or $\frac{1}{2}$.

190. Independent events. — The events of a set are **independent** if the happening of any one of the events does not affect the happening of the others.

Theorem I. — *The probability that all of a set of independent events will happen on an occasion when each of them is possible, is the product of their separate probabilities of happening.*

Proof. — **1.** For simplicity, consider a set of only two events, W and B. Suppose that any trial of W can result in u ways, in m of which W happens, and that any trial of B can result in v ways, in n of which B happens.

For concreteness think of W as the event of drawing a white ball from a bag containing u balls, of which m are white, and of B as the event of drawing a black ball from a second bag containing v balls, of which n are black. For these events, a trial of *both W and B* consists of drawing one ball from each bag. Both W and B happen if a *white* is drawn from the first bag and a *black* from the second.

2. The probability of W happening is $\frac{m}{u}$, and of B happening is $\frac{n}{v}$.

3. By Section 181, any trial of both W and B can result in uv different ways, and W and B will both happen in mn of these. Hence, the probability of both happening is $\frac{mn}{uv}$, or $\left(\frac{m}{u}\right)\left(\frac{n}{v}\right)$, which we see is the product of the probabilities of happening of the separate events.

Illustration. If the probability of A winning a certain game is $\frac{3}{10}$, and of B winning another game is $\frac{3}{5}$, then the probability that both A and B will win is $\frac{3}{5} \cdot \frac{3}{10}$, or $\frac{9}{50}$.

191. Dependent events. — If the happening of one event E_1 affects the probability of another event E_2 happening, then E_2 is said to be **dependent** on E_1.

Theorem I. — *If the probability of E_1 happening is p_1 and if, after E_1 has happened, the probability of E_2 happening is p_2, the probability of E_1 and then E_2 happening is p_1p_2.*

Proof. — **1.** Suppose that a trial of E_1 can result in u ways in m of which E_1 happens; then $p_1 = \frac{m}{u}$.

2. Suppose that any trial of E_2 can result in v different ways, and, if E_1 has already happened, that E_2 can happen in n ways; then, $p_2 = \frac{n}{v}$.

3. By Section 181, E_1 and then E_2 can happen in mn ways; E_1 can happen or fail and be followed by E_2 happening or failing, in uv ways. Hence, the probability of E_1 and then E_2 happening is $\frac{mn}{uv}$, or $\left(\frac{m}{u}\right)\left(\frac{n}{v}\right)$, which we see is p_1p_2.

Similarly, *if, after E_2 has happened, the probability of a third event E_3 happening is p_3, then the probability of E_1, and then E_2 and then E_3 happening is $p_1p_2p_3$,* and so on for any number of successive dependent events.

Illustration. Suppose that $\frac{1}{3}$ is the probability of A winning the first game of three to be played, and, if the first is won, that his proba-

bility of winning the second is $\frac{2}{5}$, and, if the second is won, that his probability of winning the third is $\frac{1}{2}$. Then, by Theorem I, A's probability of winning all three games is $\frac{1}{3} \cdot \frac{2}{5} \cdot \frac{1}{2}$, or $\frac{1}{15}$.

EXERCISE 117

1. The probability of A winning a game is $\frac{1}{3}$ and of B winning another game is $\frac{1}{5}$. Find the probability that both will win.

2. In a game where only one man can win, the probability that A will win is $\frac{1}{3}$, and that B will win is $\frac{1}{5}$. Find the probability that one or other of A and B will win.

3. The probability of A arriving in a certain town is $\frac{1}{4}$, and, if he arrives, the probability that he will meet B there is $\frac{1}{5}$. Find the probability that A will arrive and meet B.

One bag contains 4 *white and* 6 *black balls, and a second bag contains* 3 *white and* 7 *black balls. In problems* 4 *to* 8, *find the probability that the event described will occur:*

4. If 1 ball is drawn from each bag, both balls will be white.

5. If a ball is drawn from a bag selected at random, the ball will be white.

HINT. — 1. A white may be obtained (*a*) from the first bag or (*b*) from the second bag. These are two *mutually exclusive* cases.

2. *To find the probability of* (*a*) *occurring.* — The probability of selecting the first bag is $\frac{1}{2}$; after selecting it, the probability of drawing a white ball from it is $\frac{4}{10}$ or $\frac{2}{5}$.

6. If 2 balls are drawn from the first bag, and if the first drawn is replaced before the second is drawn, then both balls will be white.

7. If 2 balls are drawn in succession from the second bag, and not replaced, both balls will be black.

8. If 2 balls are drawn together from a bag selected at random, both will be black.

A bag contains 15 *white and* 10 *black balls. There is a red spot on* 10 *white and on* 5 *black balls. In problems* 9 *to* 11, *find the probability that the event described will occur:*

9. If two balls are drawn in succession, and not replaced, one will be white and one black.

10. If a ball is drawn, it will be white with a red spot.

11. If 3 balls are drawn in succession, and not replaced, at least 1 will be black.

The probability of a certain man living 10 *years is* $\frac{1}{4}$, *and of his wife living* 10 *years is* $\frac{1}{5}$. *In problems* 12 *to* 15, *find the probability that the event described will occur:*

12. Both will live 10 years.

13. The man *will* and the wife will *not* live 10 years.

HINT. — The probability that the wife will *not* live is $(1 - \frac{1}{5})$, or $\frac{4}{5}$.

14. Both will die before the end of 10 years.

15. At least one of them will live 10 years.

A deer is shot at by H *and by* K. *The probability that it will be hit by* H *is* $\frac{1}{3}$ *and by* K *is* $\frac{1}{4}$. *In problems* 16 *and* 17, *find the probability that the event described will occur:*

16. H will hit and K will miss.

17. (*a*) Both will miss; (*b*) at most one will hit.

18. The probability that a certain man will live 20 years is $\frac{1}{3}$ and that his wife will live 20 years is $\frac{2}{7}$. Find the probability (*a*) that the wife will and the man will not live 20 years; (*b*) that at most one will live 20 years.

19. To determine who is to receive a certain prize, A tosses a coin with B, the winner tosses with C, and the winner tosses with D. The winner of the last toss receives the prize. Find each person's probability of receiving the prize.

20. If a coin and a die are tossed, what is the probability of obtaining (*a*) a head, and a 6 on the die; (*b*) a tail, and at least 4 on the die?

21. One box contains 6 apples and 11 oranges, and a second box contains 3 apples and 5 oranges. If a man selects a box at random, and then draws one piece of fruit, what is the probability that an apple is drawn?

22. Show that 7 is the most probable total which can result if two dice are thrown.

23. If six persons take seats at random on a bench, what is the probability that a certain two will be together?

PROBLEMS INVOLVING SUCCESSIVE TRIALS OF AN EVENT

24. The probability of A winning, whenever he plays a certain game, is $\frac{1}{4}$. If A plays six times, what is his probability of winning four games and losing two?

Solution. — 1. Four games, to be won, can be selected from six played in ${}_6C_4$ or 15 different (mutually exclusive) ways.

2. The probability of A losing in a game is $(1 - \frac{1}{4})$, or $\frac{3}{4}$.

3. The probability that A will win any particular four and lose the other two is (Theorem I, page 282) $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{3}{4}$, or $\frac{9}{4096}$, which is the probability of the occurrence of any one of the 15 *mutually exclusive cases* in which A can win four and lose two games.

4. Hence, by Theorem I, Section 189, the probability of *one or other* of the 15 cases occurring is the sum of 15 probabilities (each equal to $\frac{9}{4096}$), or $15 \cdot \frac{9}{4096}$, or $\frac{135}{4096}$.

25. If A, of Problem 24, plays five games, find the probability that he will win (*a*) exactly three games; (*b*) at least three games.

Hint for (*b*). — He wins at least three in case he wins exactly three, or exactly four, or exactly five. Find the probability of each of these mutually exclusive cases occurring, and then use Theorem I, Section 189.

26. Prove the following theorem and corollary:

Theorem. — *If p is the probability that an event will happen, and q the probability that it will fail, at any trial, then the probability that the event will happen exactly k times out of n trials is*

$$ {}_nC_k p^k q^{n-k}. \qquad (1) $$

Corollary. — *The probability that the event will happen at least k times out of n trials is*

$$p^n + {}_nC_{n-1}p^{n-1}q + {}_nC_{n-2}p^{n-2}q^2 + \cdots + {}_nC_{n-k}p^kq^{n-k}. \quad (2)$$

HINT for the proof of the theorem. — By Theorem I, Section 190, the probability that the event will happen in any particular k trials and fail in the other $(n - k)$ trials is

$(p \cdot p \cdot \cdots$ to k factors$) \cdot (q \cdot q \cdot \cdots$ to $n - k$ factors$)$, or p^kq^{n-k}.

NOTE. — In Problem 26, notice that (1) is the same as *the term in the expansion of* $(p + q)^n$ *which involves* p^k, and that (2) is the *sum of the first* $(n - k + 1)$ *terms of* $(p + q)^n$.

27. Find the probability that, in six tosses with a coin (or, in one toss with six coins), (*a*) exactly five will fall heads; (*b*) at least five will fall heads.

28. If three dice are tossed, what is the probability (*a*) that exactly two will give aces; (*b*) that at least two will give aces?

29. A bag contains 4 white and 2 black balls. If we draw five balls in succession, replacing each one before the next is drawn, what is the probability that, of those drawn, (*a*) exactly four are white; (*b*) at least four are white?

30. In any single game, the probability that A will win is $\frac{1}{3}$. Find the probability that A will win at least five games out of seven played.

31. In a certain city, in which there are 7300 automobiles, ninety-seven were stolen in the last year. On the basis of this experience, what is the probability that a man, who owns three automobiles, will have exactly two stolen in the next year?

32. What is the probability that, of five men whose age is 30, exactly four will remain alive 20 years from now?

33. What is the probability of throwing at least three sevens in five throws with a pair of dice?

CHAPTER XXIV

DETERMINANTS

192. Introduction.* — We shall extend our previous treatment of determinants and shall consider determinants of any order. For convenience, we shall make a fresh start, by defining a determinant of any order. We shall show, later, that this definition includes our previous definitions of determinants of the 2d and 3d orders.

193. Inversions. — In considering any permutation of positive integers, we shall say there is an **inversion** whenever a greater number precedes a lesser.

Thus, in (1, 5, 3, 2, 4) there are four inversions because 5 precedes 3, 2, and 4, and 3 precedes 2.

Similarly, in any permutation of letters a, b, c, $\cdots$ we shall say there is an inversion whenever one letter *precedes* another which *follows it in alphabetical order.*

194. Auxiliary theorem. — *In any permutation of integers* (or, *letters*), *if two adjacent integers* (or, *letters*) *are interchanged, the number of inversions is increased or decreased by* 1.

Illustration. In 1, 5, 3, 2, 4, there are four inversions; on interchanging 2 and 3, we obtain 1, 5, 2, 3, 4, which has only three inversions.

Proof.— Let x and y be adjacent integers in the permutation $AxyB$, where A and B denote the groups of integers which precede, and follow xy. On interchanging x and y we obtain $AyxB$. Any inversions in A, or in B, and any due to the fact that A

* Sections 49 to 52, inclusive, should be reviewed before proceeding.

precedes x and y, or that Axy precedes B, are present in both $AxyB$ and $AyxB$. The only change in inversions is due to this: if xy is an inversion, then yx is not, and $AxyB$ has one more inversion than $AyxB$; or, if xy is not an inversion, then yx is, and hence $AxyB$ has one less inversion than $AyxB$.

The preceding method of proof obviously applies to the case of the theorem referring to letters.

195. Combined inversions of letters and subscripts; supplementary.* — In dealing with determinants, we shall represent numbers by letters with subscripts attached, as a_1, b_3, c_2, etc. Let T denote any product of such symbols in which all letters and subscripts are different. In T, let s and l be the number of *inversions in the subscripts*, and *in the letters*, respectively.

Thus, if $T = a_1b_4d_3c_2$, then $s = 3$ and $l = 1$.

Theorem I. — *If two adjacent factors in T are interchanged, then the combined number, $(s + l)$, of inversions in subscripts and in letters, is increased by $+ 2$, or by $- 2$, or by 0.*

Thus, if $T = a_1b_4d_3c_2$, $(s + l) = 3 + 1 = 4$; on interchanging d_3 and c_2, we obtain $T = a_1b_4c_2d_3$, for which $(s + l) = 2 + 0 = 2$.

Proof. — If we interchange two adjacent symbols in T, we interchange two adjacent letters and two adjacent subscripts. Hence, by Section 194, we increase s by ± 1 and l by ± 1. Therefore, we increase $(s + l)$ by $(\pm 1 \pm 1)$, which is either $+ 2$ or $- 2$ or 0.

Theorem II. — *For any product T, the combined number of inversions in letters and subscripts is either always even, or always odd, regardless of the order in which the factors of T are written.*

* If the instructor decides to omit this section, then the complete proof of Property I, page 292, should not be read.

Proof. — Any order for the factors can be obtained from any other order by successive interchanges of adjacent factors. Each interchange increases $(s + l)$ by $+ 2$, $- 2$, or 0, or, in any case, changes $(s + l)$ by an *even* number. Hence, if $(s + l)$ is *even* (or, odd) for one order of the factors, it is *even* (or, odd) for all orders.

196. Determinants of any order. — Consider a square array of n^2 numbers, called **elements**, arranged in n rows and n columns of n numbers each, and inclosed by two vertical bars. Represent each element by a letter with a subscript attached showing the number of the row where the element lies, and use the same letter for all elements in the same column.

Thus, we indicate arrays of 2^2, $(n = 2)$, and of 3^2, $(n = 3)$, elements as follows: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

We may indicate an array of n^2 elements by $\begin{vmatrix} a_1 & b_1 \cdots r_1 \\ a_2 & b_2 \cdots r_2 \\ \cdot & \cdot \ \cdot \ \cdot \ \cdot \\ \cdot & \cdot \ \cdot \ \cdot \ \cdot \\ a_n & b_n \cdots r_n \end{vmatrix}$. (1)

Definition I. — *A square array of n^2 numbers, such as* (1), *is called a* ***determinant of the nth order.*** *It is a symbol for the sum of all products that can be made*

1. *by taking as factors one and only one element from each row, and from each column, and*

2. *by giving to each such product* ***a plus or a minus sign*** *according as the number of inversions in the subscripts is* ***even or odd****, after the letters of the product are written* ***in the order in which they appear in the first row of the determinant.***

The sum of the signed products described in Definition I is called the **expansion** of the determinant.

The **principal diagonal** of a determinant is the line from the upper left- to the lower right-hand corner.

Illustration. Let D represent the adjoining determinant of order 3. We obtain all products in its expansion by attaching the subscripts 1, 2, and 3, in every possible order, to the type product abc. We thus obtain the six products $a_1b_2c_3$, $a_1b_3c_2$, $a_2b_1c_3$, $a_2b_3c_1$, $a_3b_1c_2$, and $a_3b_2c_1$. We determine their signs by Step 2 of Definition I. Thus, we give $a_2b_3c_1$ a *plus* sign because the subscripts 2, 3, and 1 show *two* inversions, and $a_2b_1c_3$ a *minus* sign because 2, 1, and 3 show *one* inversion. Finally, we obtain

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

This expression is the same as formula 2, page 64. Hence, we have proved the following statement:

Definition I *includes, as a special case, the definition of determinants of the* 3*d order, on page* 64.

Similarly, it can be shown that *Definition* I *includes, as a special case, our previous definition of determinants of the* 2*d order.*

Theorem I. — *The expansion of a determinant of order n contains n ! terms.*

Proof. — The number of terms is the same as the number of different ways in which we can attach the subscripts $1, 2, \cdots, n$ to the n letters used to denote columns. This number of ways is ${}_nP_n$ or $n!$.

Thus, the expansion of a determinant of the 4th order contains 4! or 24 terms.

Note 1. — Suppose that we use the same letter for all elements of the same row, with the row-letters in alphabetical order from top to bottom, and subscripts to distinguish the columns. Then, we should replace Step 2 of Definition I by the following statement: *give to each product a plus or a minus sign according as the inversions among the letters is even or odd, after the factors of the product are written so that the* **subscripts appear in the natural order (1, 2, 3, ···).**

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

EXERCISE 118

1. By use of Definition I, write out the expansion of the adjoining determinant of order 4.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

197. Properties of determinants. — In this section, Definition I, Section 196, will be referred to simply as Definition I.

Property I. — *The value of a determinant is not changed if corresponding rows and columns are interchanged.*

Proof. — 1. In D, suppose that the letters used to distinguish columns are in alphabetical order from left to right. Let D' be obtained by interchanging corresponding rows and columns of D. We wish to show that $D = D'$.

Illustration 1. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then $D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Here, the student can easily verify by direct expansion that $D = D'$. The continuity of the chapter is not broken by omitting the remainder of this proof. Section 195 is a prerequisite for the remainder.

2. Write out the terms of D with their factors arranged as directed in Step 2 of Definition I, and the terms of D' with their factors arranged as directed in Note 1, Section 196. Then, except perhaps for its sign, any term T' of D' can be obtained from a corresponding term T of D by rearranging the factors of T.

In Illustration 1, one term of D is $T = -a_2b_1c_3$ and the corresponding term of D' is $T' = -b_1a_2c_3$.

3. There are no inversions among the letters in T, and among the subscripts in T'. Moreover, by Theorem II, Section 195, the combined number of inversions among subscripts and letters is either *even in both* T and T', or else *odd in both*. Hence, the number of inversions among *letters* in T' and the number of inversions among *subscripts* in T, are either *both even* or *both odd*. Hence, T' has the same sign in the expansion of

D' that T has in D. Therefore, $D = D'$ because terms of D and of D' with the same factors also have the same sign.

NOTE. — From Property I, it follows that, *for every theorem concerning the columns of a determinant, there is a corresponding theorem concerning the rows.* Hence, we shall state the following properties as true for both rows and columns, but shall prove only that part referring to columns.

Property II. — *If all elements of a column (or, row) of D are multiplied by the same number k, the value of the determinant is multiplied by k.*

Proof. — One and only one element of the column is a factor of each term of D. Hence, if each element of the column is multiplied by k, we obtain a new determinant D', in whose expansion each term is k times a corresponding term of D. Therefore $D' = kD$.

Illustration 2. $k\begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} = \begin{vmatrix} 3 & 4k \\ 5 & 7k \end{vmatrix}$.

From Illustration 2, it is obvious that Property II justifies the following mechanical operation on a determinant:

Property III. — *A common factor k of all elements of a column (or, row) may be removed and written before the determinant.*

Property IV. — *If two columns (or, rows) are interchanged, the sign of the determinant is changed.*

Proof: **Part I.** — First, let us prove that *if D' is obtained by interchanging* two *adjacent columns of D, then $D = -D'$.*

Illustration 3. For concreteness, think of

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix}.$$

1. Write out the terms of D and of D' with their factors arranged as directed in Step 2 of Definition I. Apart perhaps

from sign, any term T' of D' can be obtained from a corresponding term T of D by interchanging two adjacent letters in T. This also interchanges two *adjacent* subscripts in T.

In Illustration 3, one term of D is $T = a_3b_1c_2$; the corresponding term of D' is $T' = -a_3c_2b_1$.

2. Hence, by Theorem I, Section 194, the number of inversions among the subscripts in T' is *even* or *odd* according as the number of inversions among the subscripts in T is *odd* or *even*. Therefore, by Step 2 in Definition 1, corresponding terms in D and D' have opposite signs. Hence, $D = -D'$.

Proof: **Part II. — 1.** Now, suppose U and V are any two columns of D, with U to the left of V and with h columns between U and V. We can accomplish the interchange of U and V by $(2h + 1)$ successive interchanges of adjacent columns as follows:

Starting at the left, interchange U, in succession, with each of the h columns between U and V. This brings U next to V. Then, starting at the right, interchange V, in succession, with U, and with the h columns originally between U and V.

2. By Part I, the interchange of U and V in Step 1 caused $(2h + 1)$ changes, or an *odd* number of changes in the sign of D. Hence, the interchange of U and V changes the sign of D.

Property V. — *If two columns (or, rows) are identical, the value of the determinant is zero.*

Proof. — Interchange the identical columns in D. By Property IV, the value of the new determinant is $-D$. But, since the columns were identical, the new determinant is the same as the original one. Hence, $D = -D$, or $2D = 0$, or $D = 0$.

Property VI. — *If all elements of a column (or, row) are zero, the value of the determinant is zero.*

Proof. — Each term of the expansion is zero, because each term contains one factor from the column of zeros.

Property VII. — *If each element of some column (or, row) is expressed as the sum of two numbers, the determinant can be expressed as the sum of two determinants.*

Illustration 4.
$$\begin{vmatrix} a_1 & (b_1+b_1') & c_1 \\ a_2 & (b_2+b_2') & c_2 \\ a_3 & (b_3+b_3') & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1' & c_1 \\ a_2 & b_2' & c_2 \\ a_3 & b_3' & c_3 \end{vmatrix}.$$

Proof. — Property VII is true for the case in the illustration because each term of the determinant on the left is the sum of the corresponding terms of the two determinants on the right. For instance, $a_1(b_3 + b_3')c_2 = a_1b_3c_2 + a_1b_3'c_2$. A similar relation between terms establishes Property VII in the most general case.

Similarly, if each element of some column is the sum of three numbers, the determinant can be expressed as the sum of three determinants.

Property VIII. — *The value of a determinant is not changed if to each element of any column (or, row) we add k times the corresponding element of some other column (or, row).*

Proof. — Consider the following determinants:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \qquad D' = \begin{vmatrix} a_1 & (b_1+kc_1) & c_1 \\ a_2 & (b_2+kc_2) & c_2 \\ a_3 & (b_3+kc_3) & c_3 \end{vmatrix}.$$

By use of Property VII, and then by use of Property III,

$$D' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & kc_1 & c_1 \\ a_2 & kc_2 & c_2 \\ a_3 & kc_3 & c_3 \end{vmatrix} = D + k \begin{vmatrix} a_1 & c_1 & c_1 \\ a_2 & c_2 & c_2 \\ a_3 & c_3 & c_3 \end{vmatrix}.$$

By Property V the last determinant is zero. Hence, $D' = D$, which proves Property VIII for this special case. The same method of proof may be used to establish Property VIII for determinants of any order.

198. Minors. — In any determinant, if the row and column containing a given element, say e, are blotted out, the

determinant formed from the remaining elements is called the **minor** of e.

Thus, in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, the minor of c_2 is $\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$.

199. Terms containing a_1. — *In any determinant D, if a_1 is the element in the upper left-hand corner and if A_1 is its minor, then the terms of D involving a_1 are given by a_1A_1.*

Illustration **1.** Let

$$D = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}; \text{ then } A_1 = \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}.$$

The proof will be given for the special case in Illustration 1, but the method of proof will be general.

Proof. — **1.** Any term of D containing a_1 is of the form $\pm\, a_1b_ic_jd_k$, where (i, j, k) is a permutation of (2, 3, 4). Hence, there are 6 different terms involving a_1, because there are 3! or 6 permutations of (2, 3, 4) taken all at a time.

2. Write out the terms of A_1, with factors arranged as directed by Step 2 of Definition I, and let T be any term. Then, apart perhaps from sign, a_1T is a term of D, for it has one and only one factor from each row and column of D.

Thus, $-\, b_2c_4d_3$ is a term of A_1; $-\, a_1b_2c_4d_3$ is a term of D.

3. Since the subscripts in T are a permutation of 2, 3, 4, the number of inversions in the subscripts of T is the same as in a_1T, because the subscript 1 on a_1 does not change the number of inversions. Hence, the sign of T, in A_1, is the correct sign for a_1T in the expansion of D. Therefore, since A_1 contains 3! or 6 different terms, the 6 terms of the product a_1A_1 are the 6 terms of D which involve a_1.

200. Terms involving any element. — *In any determinant D, if e is the element in the hth row and kth column, and if E is the minor of e, the terms of D involving e are $+ eE$ or $- eE$ according as $(h + k)$ is even or odd.*

Proof. — **1.** Interchange the hth row with the one above it, then with the new one above it, etc., until, after $(h - 1)$ interchanges, the original hth row becomes the 1st row. In the resulting determinant, interchange the kth column with the one at its left, then with the new column at its left, etc., until, after $(k - 1)$ interchanges, the original kth column becomes the 1st column. Call the final determinant D'. *The element e is in the upper left-hand corner of D'.*

2. In D', the minor of e is E, because on blotting out the elements of the 1st row and 1st column of D' we leave the elements of E in the order they had originally in D. Hence, by Section 199, the terms of D' involving e are given by eE.

3. D can be obtained by starting with D' and performing in reverse order the $[(h - 1)+(k - 1)]$ or $(h + k - 2)$ interchanges of Step 1. By Property IV each interchange changes the signs of all terms. Hence, since eE gives the terms of D' involving e, then the terms of D which involve e are $+ eE$ or $- eE$ according as $(h + k - 2)$ is even or odd, or (which is the same), according as $(h + k)$ is even or odd.

201. Method for expanding a determinant D by minors, according to the elements of a given column (or, row):

1. *Multiply each element of the column by its minor, and give the product a* ***plus*** *or a* ***minus*** *sign according as the sum of the numbers of the row and the column containing the element is* ***even*** *or* ***odd***.

2. *Take the sum of these signed products. This sum is D.*

Proof. — Each term in the expansion of D contains one and only one element of the given column. Hence, the expansion is the sum of the terms involving the 1st element of the column, plus the terms involving the 2d element, etc., and, by Section 200, these terms are the ones described in Step 1 of the method.

Illustration 1. Expanding according to the elements of the 1st column,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

Illustration 2. Expanding according to the elements of the 2d row,
$$\begin{vmatrix} 3 & -1 & 5 & 0 \\ -1 & 0 & -4 & 2 \\ 2 & 3 & -2 & 6 \\ 4 & -2 & -3 & -1 \end{vmatrix}$$
$$= -(-1)\begin{vmatrix} -1 & 5 & 0 \\ 3 & -2 & 6 \\ -2 & -3 & 1 \end{vmatrix} + 0 - (-4)\begin{vmatrix} 3 & -1 & 0 \\ 2 & 3 & 6 \\ 4 & -2 & -1 \end{vmatrix} + 2\begin{vmatrix} 3 & -1 & 5 \\ 2 & 3 & -2 \\ 4 & -2 & -3 \end{vmatrix}.$$

NOTE. — The signs to use in expanding by minors can be remembered by use of the adjoining diagram where the sign in each place is the one which should be attached to the minor of the corresponding element in expanding.
$$\begin{vmatrix} + & - & + & . & . & . & . \\ - & + & - & . & . & . & . \\ + & - & + & . & . & . & . \\ - & + & - & . & . & . & . \\ . & . & . & . & . & . & . \end{vmatrix}$$

202. To evaluate a determinant of the second order, use the method of Section 49. If the order of a determinant is greater than 2, it is usually most convenient to proceed as follows:

1. *By use of Property* VIII, *page* 295, *reduce all, except at most two, of the elements of a certain column* (*or, row*) *to zeros.*

2. *Expand the resulting determinant by minors, according to the elements of the column* (*or, row*) *containing the zeros. If the order of the minors is greater than* 2, *proceed with each minor as with the original determinant.*

Example 1. — Compute the value of $D = \begin{vmatrix} 5 & 7 & 8 & 6 \\ 11 & 16 & 13 & 11 \\ 14 & 24 & 20 & 23 \\ 7 & 13 & 12 & 2 \end{vmatrix}$.

Solution. — 1. Subtract the 1st row from the last; twice the 1st row from the 2d row; and three times the 1st row from the 3d row. Then, by use of Property VIII, and later by use of Property III,

$$D = \begin{vmatrix} 5 & 7 & 8 & 6 \\ 1 & 2 & -3 & -1 \\ -1 & 3 & -4 & 5 \\ 2 & 6 & 4 & -4 \end{vmatrix} = 2 \begin{vmatrix} 5 & 7 & 8 & 6 \\ 1 & 2 & -3 & -1 \\ -1 & 3 & -4 & 5 \\ 1 & 3 & 2 & -2 \end{vmatrix}.$$

2. In the last determinant, subtract the 2d row from the last, and five times the 2d row from the 1st row; add the 2d row to the 3d row; then expand the resulting determinant according to the elements of the first column:

$$D = 2 \begin{vmatrix} 0 & -3 & 23 & 11 \\ 1 & 2 & -3 & -1 \\ 0 & 5 & -7 & 4 \\ 0 & 1 & 5 & -1 \end{vmatrix} = 2 \left\{ -(1) \cdot \begin{vmatrix} -3 & 23 & 11 \\ 5 & -7 & 4 \\ 1 & 5 & -1 \end{vmatrix} \right\}$$

$$= 2(-598) = -1196.$$

In expanding by minors, there was only one term, because, when an element is zero, the product of it by its minor is also zero. The final determinant of the 3d order was computed by first reducing all but one of the elements to zeros in the last row.

Notice that the method for expanding determinants of the 3d order, given in Note 1 on page 64, does not apply to determinants of higher order.

EXERCISE 119

1. Expand the adjoining determinant by minors according to the elements of the 1st column; of the 2d column; of the 3d row. In each case write out the minors explicitly.

$$\begin{vmatrix} c_1 & m_1 & r_1 & v_1 \\ c_2 & m_2 & r_2 & v_2 \\ c_3 & m_3 & r_3 & v_3 \\ c_4 & m_4 & r_4 & v_4 \end{vmatrix}.$$

Evaluate by use of expansion by minors:

2. $\begin{vmatrix} 1 & 5 & 2 \\ 4 & 7 & 3 \\ 2 & -3 & 6 \end{vmatrix}$. **3.** $\begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 4 \\ -2 & 1 & -3 \end{vmatrix}$. **4.** $\begin{vmatrix} -2 & 0 & 2 \\ -1 & 3 & 1 \\ 4 & -2 & 3 \end{vmatrix}$.

5. $\begin{vmatrix} 3 & -1 & 4 & 2 \\ 4 & 4 & 4 & 6 \\ 8 & 5 & -8 & 4 \\ 6 & 6 & 6 & 9 \end{vmatrix}$. **6.** $\begin{vmatrix} 2 & -3 & 1 & 4 \\ 5 & 2 & 0 & 3 \\ 0 & -1 & 2 & -5 \\ 3 & -3 & 2 & 3 \end{vmatrix}$.

7. $\begin{vmatrix} 6 & -5 & 1 & 4 \\ 5 & -2 & 2 & 4 \\ 7 & -8 & -1 & 4 \\ 3 & -6 & -3 & 9 \end{vmatrix}$.

8. $\begin{vmatrix} 1 & 2 & 5 & 7 \\ -6 & 3 & 1 & 9 \\ -3 & 5 & 2 & 7 \\ 2 & 1 & 4 & 3 \end{vmatrix}$.

9. $\begin{vmatrix} 2 & 1 & 3 & 7 \\ 3 & 4 & 5 & 8 \\ 6 & 4 & -2 & -4 \\ 8 & 4 & -3 & 2 \end{vmatrix}$.

10. $\begin{vmatrix} 3 & -1 & 2 & 3 \\ 1 & 0 & 3 & 1 \\ 2 & 3 & 0 & 1 \\ 5 & 2 & 4 & -5 \end{vmatrix}$.

11. $\begin{vmatrix} 1 & x & y \\ 1 & x^2 & y^2 \\ 1 & x^3 & y^3 \end{vmatrix}$.

12. $\begin{vmatrix} 3a & c & b \\ a & 2 & 3 \\ c & b & 4a \end{vmatrix}$.

13. Expand and solve the equation $\begin{vmatrix} 1 & 2 & 3 \\ 1 & x & 3 \\ 1 & 3 & x \end{vmatrix} = 0$, and check the solution by substitution in the original determinant.

14. Without expansion, show that the function of x represented by the adjoining determinant is zero when $x = 3$. What factor has the function?

$$\begin{vmatrix} 2 & 3 & 5 \\ 2 & x & 5 \\ 3 & -1 & 2 \end{vmatrix}.$$

15. Show that the adjoining determinant has the factors $(x - y)$, $(y - w)$, and $(w - x)$, and find the expression for the determinant as a product of linear factors.

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & w \\ x^2 & y^2 & w^2 \end{vmatrix}.$$

203. An auxiliary result.

Illustration. Consider the expansion of the following determinant according to the elements of the 3d column; C_1, C_2, and C_3 are the minors of c_1, c_2, and c_3, respectively.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv c_1C_1 - c_2C_2 + c_3C_3. \tag{1}$$

If, on both sides of (1), we replace c_1, c_2, and c_3 by a_1, a_2, and a_3, respectively, we obtain

$$\begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix} \equiv a_1C_1 - a_2C_2 + a_3C_3. \tag{2}$$

Since the determinant in (2) has two columns identical, it is identically zero. Hence, $a_1C_1 - a_2C_2 + a_3C_3 \equiv 0$. In a similar fashion, the following theorem can be proved for a determinant of any order.

Theorem I. — *In the expansion of a determinant D by minors, according to the elements of a given column, if we replace each element of this column by the corresponding element of some other column, the resulting expression is identically zero.*

204. Solution of linear equations by determinants. — Any system of n linear equations in n unknowns can be written in the form

$$\text{I:} \quad \begin{cases} a_1x + b_1y + c_1z + \cdots = k_1, \\ a_2x + b_2y + c_2z + \cdots = k_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_nx + b_ny + c_nz + \cdots = k_n, \end{cases}$$

where $k_1, k_2, \cdots, k_n$ are constant terms. In any system I, let Δ represent the determinant formed from the coefficients of the unknowns.

Theorem I. — *In any system of linear equations of the form* I, *containing the same number of unknowns as equations, if the determinant of the coefficients of the unknowns is not zero* ($\Delta \neq 0$), *then the system has a single solution. In this solution, the value of any unknown can be expressed as a fraction in which*

1. *the denominator is Δ, and*

2. *the numerator is the determinant obtained if, in Δ, the column of coefficients of this unknown is replaced by the column of constant terms, $k_1, k_2, \cdots, k_n$.*

NOTE. — For simplicity in writing, we shall prove Theorem I for the special case of the following system:

$$\text{II:}\quad \begin{cases} a_1x + b_1y + c_1z + d_1w = k_1, & (1)\\ a_2x + b_2y + c_2z + d_2w = k_2, & (2)\\ a_3x + b_3y + c_3z + d_3w = k_3, & (3)\\ a_4x + b_4y + c_4z + d_4w = k_4. & (4) \end{cases}$$

It will be obvious that the methods used would apply to a system of any number of equations. All details which follow refer to system II.

For II,
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1\\ a_2 & b_2 & c_2 & d_2\\ a_3 & b_3 & c_3 & d_3\\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}. \qquad (5)$$

Let K_1, K_2, K_3, and K_4 be defined as follows:

$$K_1 = \begin{vmatrix} k_1 & b_1 & c_1 & d_1\\ k_2 & b_2 & c_2 & d_2\\ k_3 & b_3 & c_3 & d_3\\ k_4 & b_4 & c_4 & d_4 \end{vmatrix}, \qquad K_2 = \begin{vmatrix} a_1 & k_1 & c_1 & d_1\\ a_2 & k_2 & c_2 & d_2\\ a_3 & k_3 & c_3 & d_3\\ a_4 & k_4 & c_4 & d_4 \end{vmatrix}, \qquad (6)$$

$$K_3 = \begin{vmatrix} a_1 & b_1 & k_1 & d_1\\ a_2 & b_2 & k_2 & d_2\\ a_3 & b_3 & k_3 & d_3\\ a_4 & b_4 & k_4 & d_4 \end{vmatrix}, \qquad K_4 = \begin{vmatrix} a_1 & b_1 & c_1 & k_1\\ a_2 & b_2 & c_2 & k_2\\ a_3 & b_3 & c_3 & k_3\\ a_4 & b_4 & c_4 & k_4 \end{vmatrix}. \qquad (7)$$

Proof of Theorem I. — **1.** First let us show that, in any solution of system II, x, y, z, and w satisfy the equations

$$\Delta \cdot x = K_1, \quad \Delta \cdot y = K_2, \quad \Delta \cdot z = K_3, \text{ and } \Delta \cdot w = K_4. \qquad (8)$$

2. *To establish for instance, the second equation of* (8), we proceed as follows: In Δ, let B_1, B_2, B_3, and B_4 be the minors of b_1, b_2, b_3, and b_4, respectively. Multiply both sides of (1) by $-B_1$, of (2) by $+B_2$, of (3) by $-B_3$, and of (4) by $+B_4$. One obtains

$$-a_1B_1x - b_1B_1y - c_1B_1z - d_1B_1w = -k_1B_1, \qquad (9)$$
$$a_2B_2x + b_2B_2y + c_2B_2z + d_2B_2w = k_2B_2, \qquad (10)$$
$$-a_3B_3x - b_3B_3y - c_3B_3z - d_3B_3w = -k_3B_3, \qquad (11)$$
$$a_4B_4x + b_4B_4y + c_4B_4z + d_4B_4w = k_4B_4. \qquad (12)$$

Add corresponding members of (9), (10), (11), and (12) *and collect terms:* The coefficient of y is found to be

$$-b_1B_1 + b_2B_2 - b_3B_3 + b_4B_4, \qquad (13)$$

which is seen to be Δ, expanded according to the elements of its 2d column. The coefficient of x is

$$-a_1B_1 + a_2B_2 - a_3B_3 + a_4B_4,$$

which is the same as (13) except that the a's replace the b's in (13). Hence, by Theorem I, Section 203, the coefficient of x is zero. Similarly it is found that the coefficients of z and w are zero. Therefore, on adding (9), (10), (11), and (12), we obtain

$$\Delta \cdot y = -k_1B_1 + k_2B_2 - k_3B_3 + k_4B_4. \qquad (14)$$

The right member of (14) is the same as (13) except that k_1, k_2, k_3, and k_4 replace b_1, b_2, b_3, and b_4, respectively. Since (13) equals Δ, hence the right member of (14) is the expansion of the determinant K_2 obtained by replacing the second column of Δ by the constant terms k_1, k_2, k_3, and k_4. Therefore, (14) becomes $\Delta \cdot y = K_2$.

3. Similarly, to establish $\Delta \cdot z = K_3$, we should commence by multiplying both sides of (1), (2), (3), and (4) by $+C_1$, $-C_2$, $+C_3$, and $-C_4$, respectively, and so on for the other equations of (8).

4. If $\Delta \neq 0$, then, from equations (8) we obtain

$$x = \frac{K_1}{\Delta},\ y = \frac{K_2}{\Delta},\ z = \frac{K_3}{\Delta},\ \text{and}\ w = \frac{K_4}{\Delta}, \qquad (15)$$

which we desired to prove.

Supplementary Note 1. — Recognize that the equations in (8) and (15) are obtained under the implicit assumption that *there is some set of values of* (x, y, z, w) which satisfy II. Hence, the preceding proof shows that, *if there is any solution*, there is *only one*, given in (15). To complete a logical proof, it should be shown, by substitution in II, that (15) actually is a solution. This substitution is omitted here.*

Note 2. — The discussion of the solution of a general system of n linear equations in n unknowns in which $\Delta = 0$, is too complicated for

* For a convenient arrangement of this substitution, see *Dickson's* Elementary Theory of Equations, page 115.

treatment in this book.* Under certain conditions, such a system is consistent (has solutions), but usually a system is inconsistent if $\Delta = 0$. The most simple condition for inconsistency is given in the following corollary.

Corollary 1. — *A system of the form* I *is inconsistent if* $\Delta = 0$, *and if any one (or more) of the numerator determinants* K_1, K_2, K_3, $\cdots$ *is different from zero.*

Proof. — **If** system II **is** consistent and if (x, y, z, w) represents any solution, then the equations in (8) hold, *even when* $\Delta = 0$. Since $\Delta \cdot x = K_1$, it follows that, if $\Delta = 0$, then $K_1 = 0$; similarly, $K_2 = K_3 = K_4 = 0$. Hence, if $\Delta = 0$ and if any one of (K_1, K_2, K_3, K_4) is *not* zero, the system *cannot* be consistent.

205. Homogeneous equations. — A linear equation is **homogeneous** in case the constant term in it is zero. Thus, the equations in system II, page 302, are homogeneous if $k_1 = k_2 = k_3 = k_4 = 0$.

Obviously, any system of homogeneous linear equations is satisfied if each unknown is zero. In certain applications, such a solution is of no use, and hence it is sometimes called the **trivial** solution.

Theorem I. — *If a system of n homogeneous linear equations in n unknowns has a solution other than the trivial one where each unknown is zero, then* $\Delta = 0$.

Proof. — **1.** Consider system II, page 302, when each of (k_1, k_2, k_3, k_4) is zero. Then, each of (K_1, K_2, K_3, K_4) is zero because each has a column of zeros. Hence, by equations 15, page 303, if $\Delta \neq 0$, the only solution of the homogeneous system is

$$x = \frac{K_1}{\Delta} = 0, \; y = 0, \; z = 0, \text{ and } w = 0.$$

* For a complete treatment of the solution of m linear equations in n unknowns, when $m < n$, $m = n$, or $m > n$, see *Dickson*, location cited, page 120.

2. Therefore, if there *is* some solution where the unknowns are *not all zero*, it is *impossible to have* $\Delta \neq 0$.

The converse of Theorem I is true; that is, if $\Delta = 0$, then the homogeneous system has non-trivial solutions (infinitely may in fact). The proof of this fact is beyond the scope of this book.*

In case non-trivial solutions exist, they can usually be obtained by solving $(n - 1)$ of the n homogeneous equations for $(n - 1)$ of the unknowns in terms of the remaining unknown.

206. A system of linear equations containing more unknowns than equations usually has infinitely many solutions, although, under certain conditions, such a system may be inconsistent.

Consider the adjoining illustration of a consistent system. In A, the determinant of the coefficients of x and y is not zero. Hence, considering z as a constant for the moment, we can solve A for x and y in terms of z, by use of determinants. We obtain $(x = z,$ $y = z - 1)$; from these, if $z = 2$, then $x = 2$ and $y = 1$; that is, $(x = 2, y = 1, z = 2)$ is one solution of A. Similarly, corresponding to each value of z, we obtain a solution of A. That is, A has *infinitely* many solutions.

$$A. \quad \begin{cases} 3x - y + 2z = 1, \\ 2x + y - 3z = -1. \end{cases}$$

207. A system of linear equations, containing more equations than unknowns, is usually *inconsistent* but, under certain conditions, is consistent.* Consider any system of m equations in n unknowns, where $m > n$. If we can solve n of the equations for the unknowns, then, if the values obtained satisfy the other $(m - n)$ equations, which were not used in solving, the system is consistent. Otherwise the system is inconsistent.

* See footnote, page 304.

Illustration 1. Consider the adjoining system. On solving the first two equations, alone, we find that they have a single solution ($x = -3$, $y = 2$). By substitution we find that this is also a solution of the 3d equation. Hence the system is consistent and has only one solution.

$$\begin{cases} x - 2y + 7 = 0, \\ 3x + 7y - 5 = 0, \\ x + y + 1 = 0. \end{cases}$$

Any system of n linear equations in $(n - 1)$ unknowns can be written in the following form (illustrated for the case $n = 3$):

$$\begin{cases} a_1x + b_1y + c_1 = 0, \\ a_2x + b_2y + c_2 = 0, \\ a_3x + b_3y + c_3 = 0. \end{cases} \tag{1}$$

Theorem I. — *In a system of n linear equations in $(n - 1)$ unknowns, if the determinant formed from the coefficients and constant terms is not zero, the system is inconsistent.*

The proof is given for the special case of system 1, where $n = 3$.

Proof. — **1.** Consider the following system, obtained from (1) by changing c_1 to c_1z, c_2 to c_2z, and c_3 to c_3z:

$$\begin{cases} a_1x + b_1y + c_1z = 0, \\ a_2x + b_2y + c_2z = 0, \\ a_3x + b_3y + c_3z = 0. \end{cases} \tag{2}$$

2. If (1) is consistent, it has a solution ($x = h$, $y = k$). Then, (2) has a solution ($x = h$, $y = k$, $z = 1$), because (2) becomes (1) if $z = 1$. In other words, if (1) is consistent, then (2) is *a homogeneous system with a non-trivial solution*, and hence, by Theorem I of Section 205, the determinant formed from the coefficients of x, y, and z in (2) is zero. Therefore, if this determinant is *not* zero, system (1) *cannot* be consistent.

EXERCISE 120

Solve by use of determinants:

1.
$$\begin{cases} 2x - y + 2z + w = 12, \\ 2x - y + 3z - 4w = 5, \\ 5x + y + z = 6, \\ -2y + z + w = 9. \end{cases}$$

Partial Solution. — 1. We use the method of Theorem I, Section 204.

$$\Delta = \begin{vmatrix} 2 & -1 & 2 & 1 \\ 2 & -1 & 3 & -4 \\ 5 & 1 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{vmatrix} = 48; \quad K_3 = \begin{vmatrix} 2 & -1 & 12 & 1 \\ 2 & -1 & 5 & -4 \\ 5 & 1 & 6 & 0 \\ 0 & -2 & 9 & 1 \end{vmatrix} = 144; \quad z = \frac{K_3}{\Delta} = 3.$$

2. The student should complete the solution by finding the values of x, y, and w.

2. $\begin{cases} x + 4y + 3z = 4, \\ 3x + 10y + z = 18, \\ 4x + 2y + 2z = 12. \end{cases}$

3. $\begin{cases} 4x + 2y + 3z + w = 7, \\ 2x + 8y + z = 10, \\ 6x - 5z - 2w = 10, \\ 8x - y + 2z + w = 11. \end{cases}$

4. $\begin{cases} 2x + 2y - z + 3w = 9, \\ -2x + y + 3z - 2w = -12, \\ 2x - y + 2z + w = 5, \\ 3x - 2y - 2z + w = 12. \end{cases}$

5. $\begin{cases} 3x + 2y + 3z - w - 2v = 11, \\ -y - w + 3v = -5, \\ x + y + 2z - 2w + v = 2, \\ -x - y - z + 2w + v = -4, \\ 2x + 2y + 3z + w - v = 8. \end{cases}$

Prove that each of the following systems is inconsistent, by use of Corollary 1, *page* 304, *or of Theorem* I, *Section* 207.

6. $\begin{cases} x + 2y - 3z = 2, \\ 2x + y - 4z = 5, \\ x - y - z = 2. \end{cases}$

7. $\begin{cases} 3x - 5y + 3 = 0, \\ x - 2y + 5 = 0, \\ 3x + 5y + 2 = 0. \end{cases}$

8. Find three different solutions of the adjoining system. $\begin{cases} 6x - y - 2z = 1, \\ 4x + y + 3z = -1. \end{cases}$

9. Show that the adjoining system is consistent. $\begin{cases} 2x - 4y + 14 = 0, \\ 3x + 7y = 5, \\ 5x + 29y - 43 = 0. \end{cases}$

10. Has the adjoining system any non-trivial solutions? If so, find two of them.

$$\begin{cases} x - 2y + 7z = 0, \\ 4x + y + 10z = 0, \\ x + y + z = 0. \end{cases}$$

11. Given that the adjoining system in the unknowns x and y is consistent, find the value of the constant k.

$$\begin{cases} x + ky + 1 = 0, \\ 3kx + 7y - 5 = 0, \\ 4x + 5ky + 2 = 0. \end{cases}$$

12. Given that the adjoining system has non-trivial solutions, find the value of the constant k.

$$\begin{cases} kx - y + 4kz = 0, \\ 4x + 5y + kz = 0, \\ x + y + z = 0. \end{cases}$$

CHAPTER XXV

PARTIAL FRACTIONS

208. Introduction. — In elementary algebra, we learn how to reduce two or more fractions to a common denominator and to express their sum as a simple fraction. In certain problems in advanced mathematics it is necessary to perform the inverse operation; that is, it is necessary to decompose a given fraction into a sum of more simple fractions, which are called **partial fractions.**

If $f(x)$ and $g(x)$ are polynomials, then $\frac{f(x)}{g(x)}$ is called a **rational fraction**. A rational fraction is called a **proper** fraction if the degree of the numerator is less than the degree of the denominator.

In this chapter we shall discuss, without proof, methods for decomposing proper fractions into partial fractions. The proof of the general validity of these methods is omitted because such proof is beyond the scope of this text.

NOTE. — An **improper** fraction is one where the degree of the numerator is greater than or equal to the degree of the denominator. To simplify an improper fraction, one would first reduce it to a sum of a polynomial and a proper fraction.

Thus, $$\frac{x^3 + 3x^2 - 5}{x^2 + 5} = x + 3 - \frac{5x + 20}{x^2 + 5}.$$

Then, the proper fraction could be decomposed by the methods of this chapter. We shall not consider any improper fractions.

209. Case I. — *When the denominator can be expressed as a product of real linear factors, all of which are different.*

Example 1. — Resolve into partial fractions:

$$\frac{7x^2 - 23x + 10}{(3x - 1)(x - 1)(x + 2)}.$$

Solution. — 1. We shall find constants A, B, and C such that, for all values of x except those for which some denominators are zero,

$$\frac{7x^2 - 23x + 10}{(3x - 1)(x - 1)(x + 2)} = \frac{A}{3x - 1} + \frac{B}{x - 1} + \frac{C}{x + 2}. \qquad (1)$$

NOTE. — The fractions on the right side in (1) are suggested because their sum has the same denominator as the given fraction.

2. Clearing (1) of fractions we obtain

$$7x^2 - 23x + 10 = A(x - 1)(x + 2) + B(3x - 1)(x + 2) + C(3x - 1)(x - 1). \qquad (2)$$

Equation 2 is true for all values of x except possibly $x = 1$, $x = -2$, and $x = \frac{1}{3}$, for each of which some denominators in (1) are zero. Hence, since (2) is true for infinitely many values of x, it follows from Corollary 1, page 210, that (2) is true for *all* values of x, *including even* $x = 1$, $x = -2$, and $x = \frac{1}{3}$.

3. From (2), when $x = 1$: $-6 = A \cdot 0 + 6B + C \cdot 0$; $B = -1$.
when $x = -2$: $84 = 21C$; $C = 4$.
when $x = \frac{1}{3}$: $\frac{28}{9} = -\frac{14}{9}A$; $A = -2$.

4. Therefore, on substituting in (1), we obtain

$$\frac{7x^2 - 23x + 10}{(3x - 1)(x - 1)(x + 2)} = -\frac{2}{3x - 1} - \frac{1}{x - 1} + \frac{4}{x + 2}.$$

210. Case II. — *When the factors of the denominator are of the first degree, and some repeated.*

Example 1. — Resolve into partial fractions:

$$\frac{4x^3 + 16x^2 - 5x + 3}{(x - 1)^2(x + 2)^2}.$$

Solution. — 1. We shall find constants, A, B, C, and D, so that, for all values of x except those for which some denominators are zero,

$$\frac{4x^3 + 16x^2 - 5x + 3}{(x - 1)^2(x + 2)^2} = \frac{A}{(x - 1)^2} + \frac{B}{x - 1} + \frac{C}{(x + 2)^2} + \frac{D}{x + 2}. \qquad (1)$$

2. Clearing of fractions in (1), we obtain

$$4x^3 + 16x^2 - 5x + 3$$
$$= A(x+2)^2 + B(x-1)(x+2)^2 + C(x-1)^2 + D(x+2)(x-1)^2 \qquad (2)$$
$$= (B+D)x^3 + (A+3B+C)x^2 + (4A - 2C - 3D)x$$
$$+ (4A - 4B + C + 2D). \qquad (3)$$

In obtaining (3), we expanded the right side of (2) and collected corresponding powers of x.

3. Since (3) is true for infinitely many values of x, the coefficient of each power of x in (3) must equal the coefficient of that power in $(4x^3 + 16x^2 - 5x + 3)$. Hence,

$$\left.\begin{aligned} B + D &= 4, \\ A + 3B + C &= 16, \\ 4A - 2C - 3D &= -5, \\ 4A - 4B + C + 2D &= 3. \end{aligned}\right\} \qquad (4)$$

4. Solving (4) for A, B, C, and D, we obtain $A = 2$, $B = 3$, $C = 5$, and $D = 1$. Hence,

$$\frac{4x^3 + 16x^2 - 5x + 3}{(x-1)^2(x+2)^2} = \frac{2}{(x-1)^2} + \frac{3}{x-1} + \frac{5}{(x+2)^2} + \frac{1}{x+2}.$$

NOTE. — In Example 1, the values of A and C can be obtained by the method used in Section 209. Thus, placing $x = 1$ in (2), we obtain $18 = 9A$; hence $A = 2$. On placing $x = -2$ in (2), we obtain $45 = 9C$; $C = 5$. After obtaining A and C in this manner, it is easy to get B and D by using only the first and second equations of (4).

211. Case III. — *When the denominator contains quadratic factors which are not repeated.*

NOTE. — In this chapter, by a *quadratic factor* we shall mean a quadratic factor which *cannot be separated into real linear factors.*

Example 1. — Resolve into partial fractions:

$$\frac{3x^2 - x + 1}{(x+1)(x^2 - x + 3)}.$$

Solution. — 1. We shall find constants A, B, and C such that, for all values of x except those for which some denominators are zero,

$$\frac{3x^2 - x + 1}{(x+1)(x^2 - x + 3)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 - x + 3}. \qquad (1)$$

2. Clear of fractions in (1):

$$3x^2 - x + 1 = A(x^2 - x + 3) + (Bx + C)(x + 1), \tag{2}$$

or,
$$3x^2 - x + 1 = (A + B)x^2 + (B + C - A)x + (3A + C). \tag{3}$$

3. Since (3) is true for infinitely many values of x, hence

$$\left.\begin{aligned} A + B &= 3, \\ B + C - A &= -1, \\ 3A + C &= 1. \end{aligned}\right\} \tag{4}$$

4. Solving (4) for A, B, and C we obtain $A = 1$, $B = 2$, and $C = -2$.

Hence,
$$\frac{3x^2 - x + 1}{(x + 1)(x^2 - x + 3)} = \frac{1}{x + 1} + \frac{2x - 2}{x^2 - x + 3}.$$

NOTE. — In Example 1, A may be obtained by placing $x = -1$ in (2): this substitution gives $5 = 5A$; $A = 1$. Then, the first two equations of (4) give the values of B and C.

212. Case IV.* — *When the denominator contains quadratic factors some of which are repeated.*

Example 1. — Resolve into partial fractions:

$$\frac{x^4 + x^3 + 2x^2 + x - 7}{(x + 2)(x^2 + x + 1)^2}.$$

Solution. — 1. We shall determine constants A, B, C, D, and E so that, for all values of x except those which make some denominators zero,

$$\frac{x^4 + x^3 + 2x^2 + x - 7}{(x + 2)(x^2 + x + 1)^2} = \frac{A}{x + 2} + \frac{Bx + C}{(x^2 + x + 1)^2} + \frac{Dx + E}{x^2 + x + 1}. \tag{1}$$

2. Clearing of fractions in (1) and collecting like powers in the right member we obtain,

$$\begin{aligned} &x^4 + x^3 + 2x^2 + x - 7 \\ &= (A + D)x^4 + (2A + E + 3D)x^3 + (3A + B + 3D + 3E)x^2 \\ &\quad + (3A + 2B + C + 2D + 3E)x + (A + 2C + 2E). \end{aligned} \tag{2}$$

3. Equating coefficients of like powers in (2), we obtain

$$\left.\begin{aligned} A + D &= 1, \\ 2A + E + 3D &= 1, \\ 3A + B + 3D + 3E &= 2, \\ 3A + 2B + C + 2D + 3E &= 1, \\ A + 2C + 2E &= -7. \end{aligned}\right\} \tag{3}$$

* The problems in the next exercise are arranged so that Case IV may be omitted if the instructor desires.

4. The solution of (3) gives $A = 1$, $B = 2$, $C = -3$, $D = 0$, and $E = -1$.

Hence,

$$\frac{x^4 + x^3 + 2x^2 + x - 7}{(x+2)(x^2+x+1)^2} = \frac{1}{x+2} + \frac{2x-3}{(x^2+x+1)^2} - \frac{1}{x^2+x+1}.$$

213. Summary. — Any proper fraction, $\frac{N}{D}$, can be resolved into a sum of partial fractions made up as follows:

I. *If a linear factor $(ax + b)$ occurs once as a factor of D, there is a partial fraction $\frac{A}{ax+b}$, in which A is some constant.*

II. *If $(ax + b)$ occurs k times as a factor of D, there are k partial fractions $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$, where A_1, A_2, $\cdots$, A_k are constants.*

III. *If a quadratic factor $(lx^2 + mx + n)$ occurs once in D, there is a partial fraction $\frac{Ex+F}{lx^2+mx+n}$, in which E and F are constants.*

IV. *If $(lx^2 + mx + n)$ occurs k times as a factor of D, there are k partial fractions*

$$\frac{C_1x + F_1}{lx^2+mx+n} + \frac{C_2x+F_2}{(lx^2+mx+n)^2} + \cdots + \frac{C_kx+F_k}{(lx^2+mx+n)^k},$$

in which C_1, C_2, $\cdots$, C_k and F_1, F_2, $\cdots$, F_k are constants.

NOTE. — Use the method of Case 1 for finding as many of the constants as possible.

EXERCISE 121

Resolve into partial fractions:

Part A. — *Involving Cases I, II, and III*

1. $\frac{5x+16}{(x-4)(2x+1)}$.

2. $\frac{13x+18}{(3x+2)(x-4)}$.

3. $\frac{2x-5}{x^2-5x-14}$.

4. $\frac{7x-14}{x^2-3x-10}$.

5. $\frac{4y^2+14y+18}{y(y+3)^2}$.

6. $\frac{x^2-4x+3}{x(x+1)^2}$.

7. $\dfrac{x^2+1}{(x-1)^3}$.

8. $\dfrac{3a^2+10a+9}{(a+2)^3}$.

9. $\dfrac{2x^2+11x-7}{(2x-5)(x^2+2)}$.

10. $\dfrac{z^2-7z+8}{(z+2)(2z^2+5)}$.

11. $\dfrac{2x^3+8x^2+2x+4}{(x+1)^2(x^2+3)}$.

12. $\dfrac{5x^2+3x+6}{x(x^2-x+3)}$.

Part B. — *Involving Case IV*

13. $\dfrac{7-13x-4x^2}{(8x^2-2x-3)(2x+1)}$.

14. $\dfrac{5x^2+1}{(x+1)(x^2-x+1)}$.

15. $\dfrac{2x^4+3x^3+2x^2-2x+1}{x^3(x^2+1)}$.

16. $\dfrac{6x^4-10x^3+18x^2-14x}{(x^2-1)(x^2+2x-3)}$.

17. $\dfrac{4x^3-3x^2-20}{(x^2+x+4)(2x^2+2x+5)}$.

18. $\dfrac{10}{(x+4)^4}$.

19. $\dfrac{x^3+8x^2-6x-3}{(x^2+x-2)(3x^2+2x-1)}$.

20. $\dfrac{3x^2-5x+6}{(1-5x)(1+3x)^2}$.

21. $\dfrac{2x}{8x^3+1}$.

22. $\dfrac{x^3-6x^2-12x-24}{x^4-14}$.

23. $\dfrac{x^3+x^2+2}{(x^2+2)^2}$.

24. $\dfrac{2x^3+x+3}{(x^2+1)^2}$.

25. $\dfrac{2x^4+5x^3+2x^2+x+1}{(2x^2+x+1)^2(x+2)}$.

26. $\dfrac{4x^4+6x^3+3x^2-3x-3}{(x-1)(2x^2+1)^2}$.

27. $\dfrac{x^7+x^5+x^3+x}{(x^2+2)^2(x^2+3)^2}$.

28. $\dfrac{3x^3}{(x^3+1)^2}$.

CHAPTER XXVI*

THE MATHEMATICS OF INVESTMENT

214. Compound Interest. — If, at stated intervals during the term of an investment, the interest due is added to the principal and thereafter earns interest as new principal, the sum by which the original principal has increased, up to any date, is called **compound interest**. At any date, the new principal, which consists of the original principal plus the compound interest, is called the **compound amount**. The time between successive additions of interest to principal is called the **interest period.**

Illustration 1. To say that the rate is 6%, compounded semi-annually, means that the interest period is 6 months. The rate per interest period is 3%.

The formula of compound interest. *Let r, expressed as a decimal, be the rate of interest, per interest period. Let P be the original principal and let A be the compound amount to which P accumulates by the end of k interest periods. Then,*

$$A = P(1 + r)^k. \tag{1}$$

Proof of equation 1. — At the end of the 1st period, the interest due is Pr and the new principal is $(P + Pr)$ or $P(1 + r)$.

Similarly, the new principal, at the end of any period, is $(1 + r)$ times the principal on hand at the beginning of the period. By the end of k periods, the original principal P will have been multiplied k times by $(1 + r)$, or by $(1 + r)^k$. That

*Logarithms and geometric progressions will be used in this chapter, but an acquaintance with logarithms is not an absolute prerequisite.

is, the compound amount at the end of k periods is $A = P(1 + r)^k$.

Illustration 1. If the rate is 8%, compounded quarterly, the compound amount on a principal of \$300 at the end of $9\frac{1}{4}$ years is obtained from (1) with $r = .02$, $k = 37$, and $P = 300$:

$$A = 300(1.02)^{37} = 300(2.0807) = \$624.21.$$

The value of $(1.02)^{37}$ was read from Table IV, page 351.

To find P, when A is given, we solve (1) for P:

$$P = \frac{A}{(1 + r)^k} = A(1 + r)^{-k}. \qquad (2)$$

We call P the **present value** of the amount A.

Illustration 2. To find the present value of \$5000, due at the end of $4\frac{1}{2}$ years, if interest is at the rate 5%, compounded semi-annually, we use (2) with $r = .025$, $k = 9$, and $A = \$5000$:

$P = \$5000(1.025)^{-9} = \$5000(.80073) = \$4003.65$ (using Table V).

To discount an amount A, due at some future date, means *to find the present value of* A. **The discount** on A is the difference between A and its present value, or $(A - P)$.

Thus, in Illustration 2, the discount on \$5000, which is due in $4\frac{1}{2}$ years, is $(5000 - 4003.65)$ or \$996.35.

NOTE. — In using equations 1 and 2, if the rate of interest per interest period is not one of those used in the interest tables, compute the result by use of logarithms, or the binomial theorem.

EXERCISE 122

Find the compound amount and the compound interest at the end of the time:

1. \$3000 is invested for $12\frac{1}{2}$ years at 8%, compounded semi-annually.

2. \$3000 is invested for $8\frac{1}{4}$ years at 8%, compounded quarterly.

3. \$4000 is invested for 10 years at 4.8%, compounded annually.

4. Verify the entry for $(1.025)^{-6}$ in Table V by computing $1 \div (1.025)^6$, or $1 \div 1.1597$, by long division.

Find the present value of \$6000, *due at the end of* $7\frac{1}{2}$ *years, if the interest rate is as follows:*

5. 6%, compounded semi-annually.

6. 8%, compounded quarterly.

7. 8%, compounded semi-annually.

8. 5.5%, compounded semi-annually.

9. If money can be invested at 4%, compounded annually, which of the following promises to pay is worth more to you: (*a*) \$1400 to be paid at the end of 7 years, or (*b*) \$2800 to be paid at the end of 27 years?

215. Nominal and effective rates. — Under compound interest, the rate *per year* at which principal grows is called the **effective rate**. The per cent quoted in describing the type of compound interest is called the **nominal rate.**

Thus, if \$100 is invested at 8%, compounded semi-annually, the nominal rate is 8%. At the end of 1 year, the compound amount is $100(1.04)^2$, or \$108.16; the compound interest is \$8.16. Hence, during the year, principal increased at the rate of $(8.16) \div (100)$ or 8.16%, which is the effective rate.

Let j be any nominal rate, and m the number of interest periods in one year. Let i be the corresponding effective rate. Then,

$$1 + i = \left(1 + \frac{j}{m}\right)^m. \qquad (1)$$

Proof of equation 1. — The rate of interest per interest period is $\frac{j}{m}$. Hence, by equation 1, Section 214, if $P = \$1$, the amount A at the end of one year, and the compound interest I, are given by

$$A = \left(1 + \frac{j}{m}\right)^m; \quad I = \left(1 + \frac{j}{m}\right)^m - 1.$$

Since the effective rate i is the rate of increase of principal per year, $i = I \div P$. Therefore, $i = I$, because $P = 1$. Hence,

$$i = \left(1 + \frac{j}{m}\right)^m - 1, \text{ or } (1 + i) = \left(1 + \frac{j}{m}\right)^m.$$

Illustration 1. The effective rate i corresponding to the rate 6%, compounded quarterly, is obtained from (1) with $j = .06$ and $m = 4$:

$$(1 + i) = (1.015)^4 = 1.0614; \quad i = .0614. \quad \text{(Using Table IV)}$$

Illustration 2. To find the nominal rate which, if compounded quarterly, will give the effective rate 4%, use (1) with $i = .04$ and $m = 2$:

$$1.04 = \left(1 + \frac{j}{2}\right)^2; \quad 1 + \frac{j}{2} = \sqrt{1.04} = 1.0198. \quad \text{(Using logarithms)}$$

Hence, $j = 2(.0198) = .0396$, or the nominal rate is 3.96%.

EXERCISE 123

Find the effective rate corresponding to each rate described:

1. 5%, compounded semi-annually.

2. 6%, compounded semi-annually.

3. 6.6%, compounded semi-annually.

4. 8%, compounded quarterly.

5. By use of logarithms, if interest is compounded semi-annually, find the nominal rate under which the effective rate will be (*a*) 6%; (*b*) 9%.

6. Is it better to invest money at 8%, compounded quarterly, rather than at 8.2%, compounded annually?

7. By interpolation in Table IV, (*a*) find k if $(1.05)^k = 5.17$; (*b*) find r if $(1 + r)^{15} = 1.3258$.

8. If interest is compounded quarterly, find the nominal rate under which \$2350 grows to \$3750, if invested for $4\frac{1}{2}$ years.

HINT. — If r is the rate per interest period,

$$3750 = 2350(1 + r)^{19}; \quad (1 + r)^{19} = \tfrac{3750}{2350}.$$

9. How long will it take \$3500 to grow to \$4708, if invested at 8%, compounded quarterly?

HINT. — Let k be the necessary number of conversion periods. Solve the resulting equation by interpolation or by the method of Section 178.

10. How long will it take for money to double itself, if invested at 6%, compounded annually?

11. Under what rate, compounded quarterly, will \$5000 be the present value of \$9300, which is due at the end of $6\frac{3}{4}$ years?

216. Annuities. — An **annuity** is a sequence of equal periodic payments. An **annuity certain** is one whose payments extend over a fixed period of time; a **contingent annuity** is one whose payments extend over a period of time whose length depends on events whose dates of occurrence cannot be accurately foretold.

Thus, a sequence of equal payments made in purchasing a house on the installment plan, form an annuity *certain*. The premiums on a life insurance policy form a *contingent* annuity because the premiums cease at the death of the insured person.

We shall deal only with annuities certain, and, in the future, this qualifying word *certain* will be omitted. The sum of the payments of an annuity made in one year is called the **annual rent**; the time between successive payments is the **payment interval**. The time between the beginning of the first payment interval and the end of the last is called the **term** of the annuity. Unless otherwise stated, the payments of an annuity are due at the ends of the payment intervals; the first payment is due at the end of the first interval and the last payment at the end of the term of the annuity.

Thus, in the case of an annuity of \$150 per month for 15 years, the payment interval is one month, the annual rent is \$1800, and the term is 15 years.

Under a specified rate of interest, the **present value of an annuity** is the sum of the present values of all payments of the annuity. The **amount of an annuity** is the sum of the compound amounts that would be on hand at the end of the term if all payments should accumulate at interest until then from the dates on which they are due.

NOTE 1. — Consider an annuity of \$100, payable annually for 5 years, with interest at the rate 4%, compounded annually. We obtain the present value A of this annuity by adding the 2d column in the table below, and the amount S by adding the 3d column.

PAYMENT OF \$100 DUE AT END OF	PRESENT VALUE OF PAYMENT	COMPOUND AMOUNT AT END OF TERM IF PAYMENT IS LEFT TO ACCUMULATE AT INTEREST
1 year	$100(1.04)^{-1} = 96.154$	$100(1.04)^{4} = 116.99$
2 years	$100(1.04)^{-2} = 92.456$	$100(1.04)^{3} = 112.49$
3 years	$100(1.04)^{-3} = 88.900$	$100(1.04)^{2} = 108.16$
4 years	$100(1.04)^{-4} = 85.480$	$100(1.04) = 104.00$
5 years	$100(1.04)^{-5} = 82.193$	$100 = 100.00$
	(add) $A = \$445.183$	(add) $S = \$541.64$

EXERCISE 124

1. If money can be invested at 6%, compounded semi-annually, form a table, as in the preceding note, to find the present value and the amount of an annuity which pays \$100 at the end of each 6 months for 3 years.

217. Annuity formulas. — In applications, the payment interval of any annuity involved usually is found to be *the same as the interest period of the interest rate.* This is the only type of problem which we shall treat.

Consider an annuity which pays \$1 at the end of each period for n interest periods. Let $(a_{\overline{n}|}$ ***at*** $i)$ represent the present value and $(s_{\overline{n}|}$ ***at*** $i)$ the amount of this annuity when the interest rate per period is i.

Determination of formulas for $(a_{\overline{n}|}$ *at* $i)$ *and* $(s_{\overline{n}|}$ *at* $i)$. — Each entry in the 2d and 4th columns of the following table should be verified by the principles of compound interest with the aid of equations 1 and 2 of Section 214.

Payment of $1 Due at the End of	Present Value of the Payment	Time from Date of Payment to End of Term	Compound Amount at End of Term if Payment is Left to Accumulate at Interest
1 period	$(1+i)^{-1}$	$(n-1)$ periods	$(1+i)^{n-1}$
2 periods	$(1+i)^{-2}$	$(n-2)$ periods	$(1+i)^{n-2}$
etc.	etc.	etc.	etc.
$(n-1)$ per.	$(1+i)^{-(n-1)}$	1 period	$(1+i)$
n periods	$(1+i)^{-n}$	0 periods	1
	Sum $=(a_{\overline{n}|}$ at $i)$		Sum $=(s_{\overline{n}|}$ at $i)$

Hence, on adding the fourth column of the table, we obtain

$$(s_{\overline{n}|}\ at\ i) = 1 + (1+i) + \cdots \text{etc.} \cdots + (1+i)^{n-2} + (1+i)^{n-1}.$$

This is a geometric progression where the ratio $r = (1+i)$, the first term $a = 1$, and the last term $l = (1+i)^{n-1}$. Moreover,

$$(rl - a) = (1+i)^n - 1, \text{ and } (r-1) = i.$$

Hence, by use of formula 6, page 163, we obtain

$$(s_{\overline{n}|}\ at\ i) = \frac{(1+i)^n - 1}{i}. \qquad (1)$$

On adding the 2d column of the table we obtain

$$(a_{\overline{n}|}\ at\ i) = (1+i)^{-n} + (1+i)^{-n+1} + \cdots \text{etc.} \cdots + (1+i)^{-2} + (1+i)^{-1},$$

which is a geometric progression with the ratio

$$r = (1+i),\ a = (1+i)^{-n}, \text{ and } l = (1+i)^{-1}.$$

Moreover, $(rl - a) = (1+i)(1+i)^{-1} - (1+i)^{-n} = 1 - (1+i)^{-n}$.

Hence, by use of formula 6, page 163, we obtain

$$(a_{\overline{n}|}\ at\ i) = \frac{1 - (1+i)^{-n}}{i}. \qquad (2)$$

If we consider an annuity problem in which

R = the periodic payment, i = the interest rate per period, and n = the number of payments, or, the number of periods in the term,

then the present value A and the amount S will be, respectively, R times the results obtained in equations 1 and 2. That is,

$$A = R(a_{\overline{n}|} \text{ at } i); \tag{3}$$

$$S = R(s_{\overline{n}|} \text{ at } i). \tag{4}$$

Illustration 1. If an annuity pays \$150 at the end of each 3 months for $8\frac{1}{2}$ years, and if interest is at the rate 6%, compounded quarterly, the present value of the annuity is

$$A = 150(a_{\overline{34}|} \text{ at } .015) = 150(26.482) = \$3972.3. \quad \text{(Using Table VII)}$$

If interest is at the rate 7%, compounded quarterly, the amount of the annuity is $S = 150(s_{\overline{34}|} \text{ at } .0175)$. Since the rate .0175 is not used in Table VI, we use equation 1 to compute S:

$$S = 150(s_{\overline{34}|} \text{ at } .0175) = 150\,\frac{(1.0175)^{34} - 1}{.0175}.$$

On computing with the aid of 5-place logarithms, we find $S = \$6689$.

EXERCISE 125

1. In Table VI, verify the entry for $(s_{\overline{15}|} \text{ at } .03)$ by use of Table IV.

2. In Table VII, verify the entry for $(a_{\overline{20}|} \text{ at } .04)$.

3. Find the present value and the amount of an annuity paying \$500 annually for 25 years, if interest is at the rate 5%, compounded annually.

4. If you deposit \$50 at the end of each 6 months in a bank which credits interest semi-annually at the rate 4%, how much will be to your credit at the end of the 20th year?

5. In purchasing a house, a man agrees to pay \$6000 cash and \$1000 at the end of each 6 months for the next 6 years. If money is worth 8%, compounded semi-annually, what would be an equivalent cash price for the house?

6. A certain trust company credits interest on all funds semi-annually at the rate 4%. What should a man deposit now

with the trust company in order to provide his family with \$500 at the end of each 6 months for the next 12 years?

7. If money is worth 5%, compounded semi-annually, find the periodic payment of an annuity which is payable semi-annually for 25 years, if its present value is \$35,000.

8. In order to create a fund of \$2000 by the end of 10 years, what should a man deposit at the end of each 6 months in a bank which credits interest semi-annually at the rate of 4%?

9. By use of geometric progressions, as in the derivation of the formulas for $a_{\overline{n}|}$ and $s_{\overline{n}|}$, find expressions for the present value and the amount of an annuity whose annual rent is \$200 and which is payable semi-annually for 10 years, if money can be invested at 8%, compounded quarterly.

218. Amortization of a debt. — A debt, whose present value is A, is said to be **amortized** under a given rate of interest, if all liabilities as to principal and interest are discharged by a sequence of periodic payments. When the payments are equal, *as is usually the case*, they form an annuity whose present value must equal A, the original liability.

Example 1. — A man borrows \$15,000, with interest payable annually at the rate 5%. The debt is to be paid, interest as due and original principal included, by equal installments at the end of each year for 5 years. (*a*) Find the annual payment. (*b*) Form a schedule showing the progress of repayment (or **amortization**) of the principal.

Solution. —(*a*) Let \$$R$ be the annual payment. The present value of the payment annuity, at the rate 5%, compounded annually, equals \$15,000, or

$$15000 = R(a_{\overline{5}|} \text{ at } .05) \qquad \text{(Equation 3, Section 217)}$$

$$R = \frac{15000}{(a_{\overline{5}|} \text{ at } .05)} = \frac{15000}{4.32947557} = 3464.622.$$

(*b*) Amortization Schedule

Year	Outstanding Principal at Beginning of Year	Interest at 5% Due at End of Year	Annual Payment at End of Year	For Repayment of Principal at End of Year
1	$15,000.000	$ 750.000	$ 3,464.622	$ 2,714.622
2	12,285.378	614.269	3,464.622	2,850.353
3	9,435.025	471.751	3,464.622	2,992.871
4	6,442.154	322.108	3,464.622	3,142.514
5	3,299.640	164.982	3,464.622	3,299.640
Totals	$46,462.197	$2323.110	$17,323.110	$15,000.000

Example 2. — In Example 1, without using the amortization table, find the principal outstanding at the beginning of the 3d year.

Solution. — The outstanding principal is the present value of the future payments, which form an annuity of $3464.62 payable annually for three years. Their present value is $3464.62(a_{\overline{3}|} \text{ at } .05) = \$9435.03$.

EXERCISE 126

1. A loan of $5000, with interest at 6%, payable semi-annually, is to be discharged, principal and interest included, by six equal semi-annual payments, the first due in 6 months. (*a*) Find the payment. (*b*) Form an amortization schedule for the debt. (*c*) Without using the amortization schedule, find the principal unpaid at the beginning of the second year, after the payment due has been made.

2. A man deposits $10,000 with a trust company which credits interest annually at the rate 5%. The fund is to provide equal payments at the end of each year for 6 years, at the end of which time the fund will be completely exhausted. Find the annual payment.

3. A purchaser of a house owes $7500, and interest at 6% is payable annually on all sums remaining due. He wishes to discharge his debt, principal and interest included, by twelve equal annual installments, the first due at the end of one year. Find the necessary payment.

4. By interpolation in Table VII, (*a*) solve $(a_{\overline{n}|}\ at\ .04) = 8.35$, for n; (*b*) solve $(a_{\overline{10}|}\ at\ r) = 8.45$, for r.

5. A house is valued at \$10,000 cash. A purchaser agrees to pay \$1600 cash, and \$1200 at the end of each 6 months as long as necessary to amortize his debt, with interest at the rate 5%, payable semi-annually. How many \$1200 payments must he make?

HINT. — If n is the number of payments, $1200(a_{\overline{n}|}\ at\ .025) = 8400$.

6. How many annual payments of \$2000, the first due at the end of one year, must be made in amortizing a debt of \$20,000, if interest is at the rate 5%, payable annually?

7. The cash value of a farm is \$12,000. In purchasing it, \$2000 is paid cash and \$1000 at the end of each 6 months for 6 years. Under what interest rate, payable semi-annually, was the transaction executed?

8. A house is worth \$25,000 cash. A purchaser was required to pay \$12,500 cash and \$1000 at the end of each 3 months for 4 years. Under what interest rate, payable quarterly, was the transaction executed?

219. A sinking fund is a fund accumulated to pay an obligation falling due at some future date. If a fund is accumulated by investing equal periodic deposits, **the amount in the fund at any time is the amount of the annuity formed by the deposits.**

Suppose that $\$A$ is borrowed, with the agreement that interest is payable as due, and that the principal shall be paid in one installment at the end of n years. If the debtor provides for the future payment of $\$A$ by the creation of a sinking fund, he is said to discharge his debt by the **sinking fund method**. Under this method, we shall assume that the debtor makes equal deposits in his sinking fund, on the same days that he pays interest to his creditor. The debtor's periodic expense is the sum of the following items:

(*a*) **Payment to creditor, of interest due on $A.**
(*b*) **Deposit for sinking fund, which is to accumulate to $A.**

Example 1. — A man borrows $10,000 and agrees to pay interest semi-annually, at the rate 6%, and to pay the principal in one installment at the end of $2\frac{1}{2}$ years. If the debtor uses the sinking fund method, and invests his fund at 4%, compounded semi-annually, find the semi-annual expense of the debt.

Solution. — 1. Semi-annual interest payable to creditor is $300.

2. Let $\$R$ be the semi-annual payment to sinking fund. The payments form an annuity whose amount at the end of $2\frac{1}{2}$ years is $10,000. Hence,

$$10000 = R(s_{\overline{5}|}\ at\ .02);\ R = \frac{10000}{(s_{\overline{5}|}\ at\ .02)} = \$1921.58.$$

3. Semi-annual expense is (300 + 1921.58), or $2221.58.

EXERCISE 127

1. A sinking fund will be accumulated by investing $1000 at the end of each 6 months at 4%, compounded semi-annually. How much will be in the fund at the beginning of the 11th year?

2. A depreciation fund is being formed to replace a machine when it is worn out at the end of 12 years. If $500 is placed in the fund at the end of each 3 months, and if the fund accumulates at the rate 8%, compounded quarterly, what is the size of the fund at the beginning of the 5th year?

3. In order to have $10,000 in a fund at the end of 15 years, what equal deposits should be placed in the fund at the end of each six months, if the fund accumulates at 6%, compounded semi-annually?

A man borrows $50,000, *on which interest is payable semi-annually at the rate* 8%, *and the principal is due at the end of* 18 *years. The man agrees to discharge the debt by the sinking fund method. Find the semi-annual expense of the debt if the sinking fund earns the given rate, compounded semi-annually:*

4. 6%. **5.** 8%. **6.** 4%. **7.** 10%.

8. To replace a machine at a cost of $20,000 at the end of 12 years, a depreciation fund will be accumulated by investing equal

deposits at the end of each year at 5%, compounded annually. Find the annual deposit.

9. A man borrows $25,000 from a creditor who desires 8% interest, payable semi-annually. Would it be better for the debtor (*a*) to amortize the debt in 10 years by semi-annual payments, or (*b*) to discharge the debt by a sinking fund method, where the sinking fund is invested at 5%, compounded semi-annually and where the principal is payable at the end of 10 years?

10. Prove that if a debt of $\$A$, with interest at the rate i, payable annually, is amortized in n years by annual payments, the annual expense is the same as if the debt were discharged by a sinking-fund method where the sinking fund accumulates at the rate i, compounded annually.

HINT. — Eventually, insert the explicit algebraic expressions for $(a_{\overline{n}|}$ *at* $i)$ and $(s_{\overline{n}|}$ *at* $i)$.

220. Bonds. — A **bond** is a written contract to pay a definite **redemption price** $\$C$ on a specified redemption date and to pay equal **dividends** $\$D$ periodically until after the redemption date. The principal $\$F$ mentioned in the face of the bond is called the **face value** or **par value**. A bond is said to be redeemed at **par** if $C = F$ (as is usually the case), and at a premium if C is greater than F. The interest rate named in a bond is called the dividend rate. The dividend D is described in a bond by saying that it is the interest, semi-annual or otherwise, on the par value F at the dividend rate.

NOTE. — The following paragraph is an extract from a bond:

The Kansas Improvement Corporation acknowledges itself to owe and, for value received, promises to pay to bearer FIVE HUNDRED DOLLARS *on January* 1*st*, 1926, *with interest on said sum from and after January* 1*st*, 1920, *at the rate* 6% *per annum, payable semi-annually, until the said principal sum is paid. Furthermore, an additional* 10% *of the said principal shall be paid to bearer on the date of redemption.*

For this bond, $F = \$500$, $C = \$550$, and the semi-annual dividend $D = \$15$ is semi-annual interest at 6% on $500. A bond is named after its face F and dividend rate, so that the extract is from a $500, 6% bond.

Corresponding to each dividend there usually would be attached to the bond an individual coupon containing a written contract to pay $\$D$ on the proper date.

When an investor purchases a bond, the interest rate i which he receives on his investment is computed *assuming that he will hold the bond until it is redeemed.*

When a bond is sold on a dividend date, the seller takes the dividend which is due. The purchaser receives the future dividends, which form an annuity whose first payment is due at the end of one dividend interval, and whose last payment is due on the redemption date.

If an investor desires a specified yield on purchasing a bond, the price $\$P$ he should be willing to pay is given by

P = (present value of $\$C$, due on the redemption date) (1)
+ (present value of the annuity formed by the dividends),

where present values are computed at the investor's interest rate.

Example 1. — A \$100, 5% bond, with dividends payable annually, will be redeemed at 105% at the end of 15 years. Find the purchase price which will yield an investor 5%, compounded annually.

Solution. — "At 105%" means at 105% of the face value, \$100; hence, the redemption price is \$105. The annual dividend is \$5. By equation 1, the price equals the present value of \$105, due at the end of 15 years, plus the present value of an annuity of \$5 per year for 15 years. Price $= 105(1.05)^{-15} + 5(a_{\overline{15}|}\ at\ .05) = 105(.48102) + 5(10.380) = 102.41$.

EXERCISE 128

1. A \$10,000, 5% bond whose dividends are payable annually, will be redeemed at par at the end of 28 years. Find the purchase price under each of the following investor's rates, compounded annually: (*a*) 4%; (*b*) 5%; (*c*) 6%.

2. A \$1000, 5% bond, with semi-annual dividends, will be redeemed at 102% at the end of 12 years. Find the price under

each of the following investor's rates, compounded semi-annually: (*a*) 3%; (*b*) 5%; (*c*) 6%.

3. A \$500, 6% bond is redeemable at 110% at the end of 18 years and its dividends are payable semi-annually. Find the price to yield an investor 8%, compounded semi-annually.

4. A \$1000, 7% bond whose dividends are payable annually, is purchased to yield 5%, compounded annually. Find the price if the bond is redeemable at par at the end of (*a*) 10 years; (*b*) 25 years; (*c*) 50 years.

A \$1000, 6% *bond, with dividends payable July* 1 *and January* 1, *is redeemable at par on July* 1, 1925. *What should an investor pay for the bond on August* 16, 1922, *if he desires the given rate, compounded semi-annually, on his investment?*

5. 5%. **6.** 8%.

HINT. — First find the proper price on July 1, 1922, the last dividend date before August 16, 1922. Then, to this price, add simple interest on it at the investor's rate from July 1, 1922, to August 16, 1922. Theoretically, we should add compound interest, but it is the practice to use simple interest in such cases.

7. A \$100, 6% bond, with annual dividends, is redeemable at par at the end of 10 years. If a purchaser buys the bond for \$112, find the yield he obtains on his investment.

HINT. — The yield is less than 6%, because the purchaser pays more than \$100. Compute the prices of the bond if sold to give the investor the following yields, compounded annually, and when you reach two successive prices, one less and the next more than \$112, then determine the yield for the \$112 price by interpolation: 6%, 5%, 4%, 3%.

8. A \$1000, 7% bond, with annual dividends, is redeemable at par at the end of 10 years. If a purchaser buys the bond for \$1140, find the yield he obtains on his investment.

NOTE. — In this chapter, the problems were limited to an extremely simple variety and many extensions of the theory and of the applications were not even mentioned. For a complete treatment, the reader is referred to textbooks on the mathematics of investment.*

* See, for instance, THE MATHEMATICS OF INVESTMENT, by W. L. Hart; D. C. Heath and Company, publishers.

CHAPTER XXVII

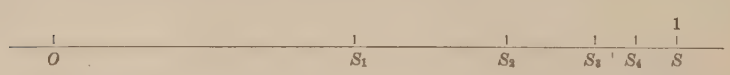
INFINITE SERIES *

221. Limit of a sequence. — Let S_1, S_2, S_3 $\cdots$ S_n, $\cdots$ be the terms of an unending sequence of numbers. To say that *a certain number S is the* **limit** *of S_n as n becomes infinite,* means that *if any positive number d is assigned, then, for all terms S_n beyond a certain place in the sequence, the absolute value of the difference $(S - S_n)$ is less than d.*

To abbreviate the statement *the limit of S_n, as n becomes infinite, is S,* we write "$\lim_{n\to\infty} S_n = S$."

Illustration 1. Consider the sequence $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, $\frac{15}{16}$ $\cdots$, where $S_n = \left(1 - \frac{1}{2^n}\right)$. In this sequence, as n grows larger and larger without limit, $\frac{1}{2^n}$ approaches zero, and S_n will be as near to 1 as may be desired for all values of n which are sufficiently large. The preceding statements prove that $\lim_{n\to\infty} S_n = 1$.

The sequence just considered can be represented by points on the accompanying scale, where 0 is zero and S is 1. Thus, $S_1 = \frac{1}{2}$, and S_1 is represented by the point halfway from 0 to S; $S_2 = \frac{3}{4}$ is the point half-



way from S_1 to S, etc. The fact that $\lim_{n\to\infty} S_n = 1$, is shown geometrically by the fact that, as we plot the successive terms of the sequence, the points approach closer and closer to the point $S = 1$.

* Section 122 is a prerequisite for this chapter.

222. Convergence of an infinite series. — If $u_1, u_2, u_3, \cdots, u_n, \cdots$ is an unending sequence of terms, then, the expression

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots, \tag{1}$$

is called an **infinite series.**

Let $S_1 = u_1$; $S_2 = u_1 + u_2$; in general, let S_n represent the sum of the first n terms in (1); then

$$S_n = u_1 + u_2 + \cdots + u_n. \tag{2}$$

Illustration 1. Consider the infinite series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots. \tag{3}$$

By addition, we obtain $S_1 = \frac{1}{2}$; $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$; $S_3 = \frac{7}{8}$; etc. In general, by use of formula 1, page 168, for the sum of a geometric progression, we obtain

$$S_n = \tfrac{1}{2} + \tfrac{1}{4} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}. \tag{4}$$

From Section 221, it is seen that $\lim_{n \to \infty} S_n = 1$.

Illustration 2. Consider the series $2 + 4 + \cdots + 2n + \cdots$. Here, $S_1 = 2$, $S_2 = 6$, $S_3 = 12$. By the formula for the sum of an arithmetic progression, $S_n = 2 + 4 + \cdots + 2n = n(1 + n)$. In this case, if n grows larger and larger, without limit, then, S_n grows larger and larger without limit; hence S_n does not approach any limit as n becomes infinite.

Illustration 3. Consider the series $1 - 1 + 1 - 1 + \cdots$. For this series, $S_1 = 1$; $S_2 = 1 - 1 = 0$; $S_3 = 1$; etc. Hence, as n becomes infinite, S_n does not approach a limit because the terms of the sequence $S_1, S_2, S_3, \cdots$ oscillate between 0 and 1.

The preceding illustrations show that *the sum of the first n terms of an infinite series may or may not approach a limit as n becomes infinite.*

Definition of convergence. — *An infinite series is said to converge, if the sum of the first n terms approaches a limit as n becomes infinite. If S is this limit, that is, if* $\lim_{n \to \infty} S_n = S$, *then we call S the sum of the infinite series.*

Hereafter, the word *series* will refer to an *infinite series*.

If a series does *not converge*, we say that it **diverges**. We say that a series is **convergent** or **divergent**, according as it converges or diverges. If S is the sum of a convergent series, we say that the series **converges to S.**

Thus, the series in Illustration 1 converges to the sum $S = 1$, while the series in Illustration 2 and the one in Illustration 3 diverge.

223. Necessary condition for convergence. — *If a series converges, then the nth term approaches zero as n becomes infinite.*

Proof. — Let $u_1 + u_2 + \cdots$ be the series. Then

$$S_n = S_{n-1} + u_n, \text{ or } u_n = S_n - S_{n-1}.$$

If S is the sum of the series, then both S_n and S_{n-1} approach S, and hence the difference $(S_n - S_{n-1})$, or u_n, approaches $(S - S)$ or zero, as n becomes infinite.

NOTE 1. — By the preceding theorem, **if the nth term of a series does not approach zero as $n \to \infty$, then the series diverges**; but, if the nth term *does* approach zero, this is *not sufficient* to insure the convergence of the series. Thus, $1 + 2 + 3 + \cdots + n + \cdots$ *diverges*, because $u_n = n$, which does not approach zero as $n \to \infty$. The nth term in $1 + \frac{1}{2} + \cdots + \frac{1}{n} + \cdots$ *does* approach zero, but, as will be proved later, this series also diverges.

224. The infinite geometric series. — In Section 122, we proved that the infinite series $a + ar + ar^2 + \cdots$ converges if $|r| < 1$. By Section 223, this series diverges if $|r| \geqq 1$, because, in this case, the nth term, ar^{n-1}, does not approach zero as n becomes infinite.

225. Determination of the sum. — By the definition of convergence, if a series converges to the sum S, then *the sum, S_n, of a large number of terms is approximately equal to S.*

Moreover, this approximation to S can be made as close as we desire if enough terms are included.

NOTE. — The sum of an infinite series is *not a sum* in the ordinary sense of the word, but it is the limit of the sequence of partial sums $S_1, S_2, S_3, \cdots$. In defining the sum of a series, we have not introduced some supernatural way of adding infinitely many terms.

Illustration 1. Let us obtain an approximation to the sum of the following convergent series:

$$1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots + \frac{1}{(2n-2)!} + \cdots. \qquad (1)$$

$1 = \quad = 1.00000. \quad S_1 = 1 \quad = 1.00000$

$\frac{1}{2!} = \frac{1}{2} \quad = .50000. \quad S_2 = 1 + \frac{1}{2!} \quad = 1.50000$

$\frac{1}{4!} = \frac{1}{24} \quad = .04167. \quad S_3 = 1 + \frac{1}{2!} + \frac{1}{4!} \quad = 1.54167$

$\frac{1}{6!} = \frac{1}{720} \quad = .00139. \quad S_4 = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} \quad = 1.54306$

$\frac{1}{8!} = \frac{1}{40320} = .00002. \quad S_5 = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \frac{1}{8!} = 1.54308$

It is reasonably certain that, in (1), terms beyond $\frac{1}{8!}$ will be so small that the sum of any number of them will not affect any partial sum S_n by more than .00001. Hence we are reasonably certain that, to four decimal places, the sum of the series is $S = 1.5431$.

NOTE. — By the method of Illustration 1, we can determine as close an approximation as we desire to the sum of a convergent series, even though no formula for the sum is known.

226. The binomial series. — By the Binomial Theorem, for a positive integral exponent, n,

$$(y + x)^n = y^n + ny^{n-1}x + \frac{n(n-1)}{1 \cdot 2} y^{n-2}x^2 + \cdots (to\ n + 1\ terms). \qquad (1)$$

The following infinite series is called the **binomial series**:

$$1 + kx + \frac{k(k-1)}{1 \cdot 2} x^2 + \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3} x^3 + \cdots. \qquad (2)$$

If k is not a positive integer, (2) contains infinitely many non-vanishing terms. If n is a positive integer, and if we place $k = n$, in (2), the series contains only $(n + 1)$ terms which are not zero, because, beyond the $(n + 1)$st term, each numerator has zero as one factor. Moreover, these $(n + 1)$ terms are exactly those obtained if in (1) we place $y = 1$. That is, *if k is a positive integer, series 2 reduces to the expansion of* $(1 + x)^k$. Thus, if $k = 3$, in (2), we obtain

$$1 + 3x + \frac{3(3-1)}{1 \cdot 2} x^2 + \frac{3(3-1)(3-2)}{1 \cdot 2 \cdot 3} x^3 + \frac{3(3-1)(3-2)(3-3)}{1 \cdot 2 \cdot 3 \cdot 4} x^4 + \cdots$$

or, simply $(1 + 3x + 3x^2 + x^3 + 0 + 0 + \cdots)$, which is $(1 + x)^3$.

If n is any real number, the following result can be proved by the methods of advanced mathematics.

Theorem I. — *If x has any value for which the binomial series 2 converges, then the sum of the series is* $(1 + x)^k$. *Moreover, for all values of k,* (2) *converges if* $|x| < 1$ and diverges if $|x| > 1$.

Example 1. — By use of (2), find $\sqrt[3]{1.1}$ to four decimal places.

Solution. — 1. $\sqrt[3]{1.1} = (1 + .1)^{\frac{1}{3}}$. Hence, to compute $\sqrt[3]{1.1}$, we substitute $x = .1$ and $k = \frac{1}{3}$ in (2) and determine the sum of the series, to four decimal places. On substituting in (2) and simplifying, we obtain $1 + .03333 - .00111 + .00006 - \cdots$. Here, $S_3 = 1.03222$ and $S_4 = 1.03228$. Hence it is reasonably certain that, to four places, $\sqrt[3]{1.1} = 1.0323$. It can be verified by logarithms that this is the correct value to four decimal places. The use of the binomial series in this case was valid because $|.1| < 1$ (see Theorem I).

EXERCISE 129

It can be proved by later methods that each of the following series converges. In each series, determine the law of formation of successive terms by inspection.

Compute the sum of each series, using enough terms to obtain the specified accuracy:

1. $1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$; correct to two decimal places.

2. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$; correct to two decimal places.

3. $1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots$; correct to four decimal places.

4. $1 - \frac{1}{2^5} + \frac{1}{3^5} - \frac{1}{4^5} + \cdots$; correct to three decimal places.

By use of the binomial series, compute each indicated power or root correct to three decimal places:

5. $(1 - .1)^{\frac{1}{2}}$. **6.** $(1.02)^{\frac{1}{5}}$. **7.** $(1.05)^{\frac{1}{3}}$. **8.** $\sqrt[4]{1.1}$.

9. $\sqrt{30}$. **10.** $\sqrt{84}$. **11.** $\sqrt[3]{120}$.

HINT for Problem 9. — The perfect square nearest to 30, is 36; $30 = 36 - 6$. Hence, $\sqrt{30} = \sqrt{36 - 6} = \sqrt{36}\sqrt{1 - \frac{6}{36}} = 6(1 - \frac{1}{6})^{\frac{1}{2}}$. Compute $(1 - \frac{1}{6})^{\frac{1}{2}}$ by the binomial series.

By use of the binomial series, obtain the following formulas:

12. $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{2 \cdot 4}x^2 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^4 + \cdots$.

13. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$.

14. $\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \cdots$.

15. $\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$.

16. Substitute $x = -.1$ in the result of problem 13, and find the sum of the series to three decimal places. Verify that the result equals $1 \div .9$.

TESTS FOR CONVERGENCE; SERIES WITH POSITIVE TERMS

227. Fundamental principle. — *Let* S_1, S_2, $\cdots$, S_n $\cdots$ *be an unending sequence of numbers which satisfy the following conditions* 1 *and* 2 *for all values of* n*:*

1. $S_n \leqq S_{n+1}$.
2. There exists a constant A such that $S_n \leqq A$.

Then, S_n approaches a limit as n becomes infinite, and this limit is not greater than A.

The proof of this principle * is given in advanced mathematics. We had an instance of its truth in Illustration 1, of Section 221. In that case, for all values of n, $S_n \leqq 1$. In colloquial language, the principle states that, *if S_n never decreases, as n becomes infinite, and always remains less than a fixed number A, then there exists a limit of S_n as n becomes infinite.*

Corollary 1. — *If $u_1 + u_2 + \cdots$ is a series of positive terms, and if there exists a number A such that, for all values of n, the sum of n terms, S_n, is not greater than A, then the series converges and its sum S is not greater than A.*

Proof. — By definition, $S_{n+1} = S_n + u_{n+1}$. Since all terms $u_1, u_2, \cdots$ are positive, hence $S_n < S_{n+1}$. Moreover, by assumption, $S_n \leqq A$ for all values of n. Hence, by the fundamental principle, the sequence $S_1, S_2, \cdots$ approaches a limit S, and $S \leqq A$. That is, the given series converges, and its sum S is not greater than A.

228. Comparison tests for convergence and divergence. Consider two series of positive terms, $u_1 + u_2 + u_3 + \cdots$ (1)

and $v_1 + v_2 + v_3 + \cdots$ (2)

Theorem *A*. — *If* (2) *converges and if each term of* (1) *is less than or equal to the corresponding term of* (2), *then* (1) *converges.*

Proof. — **1.** Let $S_n = u_1 + u_2 + \cdots + u_n$;

$S'_n = v_1 + v_2 + \cdots + v_n$.

* Sometimes, advanced theory is developed using the principle as an axiom.

2. By hypothesis, S'_n has a limit, S', as $n \to \infty$. Since all terms of (2) are positive, S'_n continually increases and hence $S'_n \leqq S'$ for all values of n. By hypothesis, $u_1 \leqq v_1$, $u_2 \leqq v_2$, etc. Therefore, for all values of n, $S_n \leqq S'_n$, and hence $S_n \leqq S'$. By Corollary 1, Section 227, it follows that (1) converges.

Theorem *B*. — *If* (1) *diverges, and if each term of* (2) *is greater than or equal to the corresponding term of* (1), *then* (2) *diverges.*

Proof. — Under our present hypotheses, if (2) were convergent, then it would follow from Theorem *A* that (1) also is convergent. Since this would contradict our hypothesis, it follows that (2) cannot converge, and therefore it diverges.

Example 1. — Prove convergent:

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{n^n} + \cdots. \tag{3}$$

Solution. — The series $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \cdots$ (4)

converges, because it is a geometric series whose ratio is $\frac{1}{2}$. Each term of (3) is less than or equal to the corresponding term of (4), because, for all values of n, $n^n \geqq 2^{n-1}$. Hence, by Theorem *A*, (3) converges.

229. Summary of series for comparison purposes:

Convergent series:

$$a + ar + ar^2 + \cdots \quad (\textit{when } r < 1) \tag{1}$$

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \quad (\textit{when } p > 1) \tag{2}$$

Divergent series:

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots \tag{3}$$

$$a + ar + ar^2 + \cdots \quad (\textit{when } r \geqq 1) \tag{4}$$

Proof that (3) diverges. — Consider the inequalities

$$1 + \frac{1}{2} > 1,$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}, \tag{5}$$

$$\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}>\frac{1}{16}+\frac{1}{16}+\cdots+\frac{1}{16}=\frac{1}{2},$$

. . . . . etc. . . . . .

On adding corresponding sides of the preceding inequalities, including all inequalities down to a certain one, we obtain

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}>1+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}, \tag{6}$$

where n depends on how many inequalities were included. The right side in (6) can be made as large as we please, by including enough inequalities, and hence the sum $1+\frac{1}{2}+\cdots+\frac{1}{n}$ grows large without limit as $n\to\infty$. Therefore (3) diverges.

Note. — The series 3 is called the **harmonic series** because its terms form a *harmonic progression.*

Proof that (2) converges. — Consider the inequalities

$$1\leqq 1,$$

$$\frac{1}{2^p}+\frac{1}{3^p}\leqq\frac{1}{2^p}+\frac{1}{2^p}=\frac{2}{2^p}=\frac{1}{2^{p-1}},$$

$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\leqq\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}=\frac{4}{4^p}=\frac{1}{4^{p-1}}=\left(\frac{1}{2^{p-1}}\right)^2,$$

$$\frac{1}{8^p}+\frac{1}{9^p}+\cdots+\frac{1}{15^p}\leqq\left(\frac{1}{2^{p-1}}\right)^3,$$

. . . . . etc. . . . . .

On adding corresponding sides of the inequalities, including all down to a certain point, we obtain

$$1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}\leqq 1+\frac{1}{2^{p-1}}+\left(\frac{1}{2^{p-1}}\right)^2+\cdots, \tag{7}$$

where the value of n depends on how many inequalities were added. The right side of (7) is the sum of a certain number of terms from the infinite geometric series.

$$1+\frac{1}{2^{p-1}}+\left(\frac{1}{2^{p-1}}\right)^2+\cdots\quad\left(\textit{whose ratio is } r=\frac{1}{2^{p-1}}\right).$$

Since $p>1$, then $r<1$, and this geometric series converges to a certain sum S. Hence, the right side in (7) is less than or equal to S, or, the sum of any number of terms in (2) is at most equal to S. Hence, by Corollary 1, Section 227, series 2 converges.

Example 1. — Prove divergent:

$$3 + 2 + 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots.$$

Solution. — 1. The convergence or divergence of a series is not affected by dropping off (or, adding on) a finite number of terms, because this merely subtracts a fixed constant from the value of S_n, for all values of n. Hence, in this example, for convenience, drop off the first two terms, obtaining the series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \cdots. \qquad (8)$$

2. Each term of (8) is greater than or equal to the corresponding term of (3), because $\frac{1}{\sqrt{n}} \geqq \frac{1}{n}$, for all values of n. Hence, by Theorem *B*, Section 228, since (3) diverges, then (8) diverges. Therefore, the original series of Example 1 diverges.

Any series of positive terms which has been proved to converge (or, to diverge) may be used for comparison purposes in establishing the convergence (or, divergence) of other series.

EXERCISE 130

Prove that each series converges:

1. $\frac{1}{5} + \frac{1}{5 \cdot 3} + \frac{1}{5 \cdot 3^2} + \cdots.$

2. $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots.$

3. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$

4. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots.$

5. $1 + \frac{1}{2!} + \frac{1}{3!} + \cdots.$

6. $\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots.$

NOTE. — It is sometimes possible to use the result of Note 1, Section 223, to prove that a series diverges.

Prove that each series diverges:

7. $2 + 4 + 6 + 8 + \cdots.$

8. $1 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \cdots.$

9. $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots.$

10. $1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \cdots.$

11. $1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots (p < 1).$

12. $\frac{2}{1} + \frac{2^2}{2} + \frac{2^3}{3} + \cdots.$

Prove that each series converges, or that it diverges:

13. $\frac{5}{\sqrt{2}} + \frac{10}{\sqrt{3}} + \frac{15}{\sqrt{4}} + \cdots$.

14. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots$.

15. $\frac{1}{1} + \frac{1}{2 \cdot 2!} + \frac{1}{3 \cdot 3!} + \cdots$.

16. $\frac{3}{1} + \frac{3^2}{\sqrt{2}} + \frac{3^3}{\sqrt{3}} + \cdots$.

230. Ratio Test. — Let $u_1 + u_2 + \cdots + u_n + \cdots$ (1)

be a series of positive terms. Form the **test ratio** $\frac{u_{n+1}}{u_n}$, and suppose that

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = R. \qquad (2)$$

NOTE. — We shall not consider series where the test ratio does not approach a limit as $n \to \infty$, although such series exist.

The Ratio Test. — **A.** *If* $R < 1$, *the series converges.*

B. *If* $R > 1$, *the series diverges.*

C. *If* $R = 1$, *the test fails.*

Proof of Part A. — **1.** Select any number r between R and 1; $R < r < 1$. Since the test ratio approaches R as a limit, hence, if n is sufficiently large, $\frac{u_{n+1}}{u_n}$ will be so near to R that it will be less than r, or, if n is sufficiently large,

$$\frac{u_{n+1}}{u_n} < r. \qquad (3)$$

2. If N is any value of n for which (3) is true, then,

$$\frac{u_{N+1}}{u_N} < r, \quad \text{or} \quad u_{N+1} < ru_N,$$

$$\frac{u_{N+2}}{u_{N+1}} < r, \quad \text{or} \quad u_{N+2} < ru_{N+1} < r^2u_N,$$

$$\frac{u_{N+3}}{u_{N+2}} < r, \quad \text{or} \quad u_{N+3} < ru_{N+2} < r^3u_N.$$

Similarly, each term of $u_{N+1} + u_{N+2} + u_{N+3} + \cdots$ (4)

is less than the corresponding term of

$$ru_N + r^2u_N + r^3u_N + \cdots. \qquad (5)$$

Since (5) is a geometric series whose ratio is r, and since $r < 1$, hence (5) converges. Therefore, by the comparison test, (4) converges. But, (4) can be obtained by dropping the terms $u_1 + u_2 + \cdots + u_N$ from (1). Hence, since (4) converges, (1) also converges.

Part B. — If $R > 1$, then (1) *diverges.*

Proof. — **1.** Select any number r between R and 1: then $1 < r < R$. Hence, for n sufficiently large, $\frac{u_{n+1}}{u_n}$ will be so near to R that it is greater than r. It is left to the student to complete the proof by a method analogous to that used in Part *A*.

NOTE. — If $\frac{u_{n+1}}{u_n}$ grows large without limit, as $n \to \infty$, we shall say that $R = +\infty$. We shall think of this case as being included in Part *B* under $R > 1$, because it is easily verified that the proof of Part *B* still holds, if $R = +\infty$.

Part C. — If $R = 1$, we cannot draw any conclusion because there are both convergent and divergent series for which $R = 1$.

Thus, by the methods used in succeeding problems, it can be shown that $R = 1$ for the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$, which diverges, and also that $R = 1$ for the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$, which converges.

Auxiliary Example 1. — Find $\lim_{n \to \infty} \frac{2n^2 + 3n + 2}{3n^2 + 5n + 1}$.

Solution. — 1. Dividing both numerator and denominator of the fraction by n^2 (because n^2 is the highest power in numerator or denominator), we obtain

$$\lim_{n \to \infty} \frac{2n^2 + 3n + 2}{3n^2 + 5n + 1} = \lim_{n \to \infty} \frac{2 + \frac{3}{n} + \frac{2}{n^2}}{3 + \frac{5}{n} + \frac{1}{n^2}}. \tag{6}$$

In the final fraction, each of $\frac{3}{n}$, $\frac{2}{n^2}$, $\frac{5}{n}$, and $\frac{1}{n^2}$ approaches zero as $n \to \infty$. Hence, from (6),

$$\lim_{n\to\infty} \frac{2n^2 + 3n + 2}{3n^2 + 5n + 1} = \frac{2 + 0 + 0}{3 + 0 + 0} = \frac{2}{3}.$$

Illustration 1. Consider the series $\frac{1}{2} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \cdots$.

In this series, $u_n = \frac{n^2}{2^n}$; $u_{n+1} = \frac{(n+1)^2}{2^{n+1}}$; $\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}$.

Hence, $R = \lim_{n\to\infty} \frac{2^n(n+1)^2}{2^{n+1} \cdot n^2} = \lim_{n\to\infty} \frac{(n+1)^2}{2n^2} = \lim_{n\to\infty} \frac{n^2 + 2n + 1}{2n^2}$.

On dividing both numerator and denominator by n^2, we find

$$R = \lim_{n\to\infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{2} = \frac{1 + 0 + 0}{2} = \frac{1}{2}.$$

Hence, since $R < 1$, the series converges.

Illustration 2. Consider the series $\frac{1}{3} + \frac{2!}{3^2} + \frac{3!}{3^3} + \cdots$.

For this series, $u_n = \frac{n!}{3^n}$; $u_{n+1} = \frac{(n+1)!}{3^{n+1}}$ and

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} = \frac{1 \cdot 2 \cdots n \cdot (n+1)}{1 \cdot 2 \cdots n} \cdot \frac{1}{3} = \frac{n+1}{3}.$$

Hence, $R = \lim_{n\to\infty} \frac{n+1}{3} = +\infty$. Since $R > 1$, the series diverges.

Illustration 3. Consider the series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$.

Here, $\frac{u_{n+1}}{u_n} = \frac{1}{n+1} \div \frac{1}{n} = \frac{n}{n+1}$. Hence, $R = \lim_{n\to\infty} \frac{n}{n+1}$.

Dividing both numerator and denominator by n, we obtain,

$$R = \lim_{n\to\infty} \frac{1}{1 + \frac{1}{n}} = \frac{1}{1 + 0} = 1.$$

Since $R = 1$, the ratio test gives no information concerning the harmonic series, although (as proved previously) it diverges.

EXERCISE 131

Test for convergence by use of the ratio test:

1. $\frac{1}{3}+\frac{2}{3^2}+\frac{3}{3^3}+\cdots$.

2. $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots$.

3. $2+\frac{2^2}{2!}+\frac{2^3}{3!}+\cdots$.

4. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\cdots$.

5. $1+\frac{1}{3}+\frac{1}{5}+\cdots$.

6. $1+\frac{2^2}{2}+\frac{2^3}{3}+\cdots$.

7. $\frac{1}{2}+\frac{2!}{2^2}+\frac{3!}{2^3}+\cdots$.

8. $5+\frac{5^2}{2!}+\frac{5^3}{3!}+\cdots$.

9. $\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots+\frac{1}{(2n)!}+\cdots$.

10. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}+\cdots$.

11. $1+2r+3r^2+\cdots$ (*where* $0<r<1$).

12. $1+\frac{2}{3!}+\frac{3}{5!}+\cdots+\frac{n}{(2n-1)!}+\cdots$.

SERIES WITH POSITIVE AND NEGATIVE TERMS

231. Alternating series. — An **alternating series** is one whose terms are alternately positive and negative.

Theorem I. — *An alternating series converges if the absolute value of each term is less than that of the preceding one and if the nth term approaches zero, as n becomes infinite.*

Proof. — **1.** Let the series be $u_1-u_2+u_3-\cdots$, where u_1, u_2, u_3, $\cdots$ are positive and $u_1>u_2>u_3>\cdots$. If $2k$ is any even integer, then

$$S_{2k}=(u_1-u_2)+(u_3-u_4)+\cdots+(u_{2k-1}-u_{2k}), \qquad (1)$$

and,
$$S_{2k}=u_1-(u_2-u_3)-(u_4-u_5)-\cdots-u_{2k}. \qquad (2)$$

Since $u_1>u_2$, hence, $(u_1-u_2)>0$. Similarly, in (1) and (2), each difference in parentheses is positive. Thus, we see from (1) that S_{2k} continually increases as k increases and, from (2), that $S_{2k}<u_1$ for all values of k. Therefore, by the principle of Section 227, S_{2k} approaches a limit S as k becomes infinite.

2. Consider S_{2k+1}, the sum of an odd number of terms; $S_{2k+1} = S_{2k} + u_{2k+1}$. By hypothesis, u_{2k+1} approaches zero as k becomes infinite. Therefore,

$$\lim_{k\to\infty} S_{2k+1} = \lim_{k\to\infty} (S_{2k} + u_{2k+1}) = S + 0 = S.$$

3. Hence, the series converges to the sum S because the sum of an odd or of an even number of terms approaches the limit S as the number of terms added becomes infinite.

To apply the test of Theorem I, it must be shown that

$$u_n > u_{n+1} \text{ and that } \lim_{n\to\infty} u_n = 0.$$

232. Theorem. — *A series of real terms converges if the series formed from the absolute values of the terms converges. That is, a series* $u_1 + u_2 + u_3 + \cdots$ *converges if the following series converges:*

$$|u_1| + |u_2| + |u_3| + \cdots. \qquad (1)$$

Proof. — **1.** Let $S_n = u_1 + u_2 + \cdots + u_n$. In S_n, let P_n be the sum of the positive terms and Q_n the sum of the absolute values of the negative terms. Then $S_n = P_n - Q_n$.

2. Let $S'_n = |u_1| + |u_2| + \cdots + |u_n|$. Then,

$$S'_n = P_n + Q_n.$$

3. By hypothesis, (1) converges to a sum S'. Since terms in (1) are positive or zero, hence $S'_n \leqq S'$, or $(P_n + Q_n) \leqq S'$.

4. Therefore, for all values of n, $P_n \leqq S'$ and $Q_n \leqq S'$. Moreover, as n increases, P_n and Q_n never decrease. Hence, by the principle of Section 227, P_n approaches a limit P and Q_n a limit Q, as $n \to \infty$. Since $S_n = P_n - Q_n$, therefore,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (P_n - Q_n) = P - Q.$$

Hence, $u_1 + u_2 + u_3 + \cdots$ converges, and its sum is $(P - Q)$.

233. Extension of the ratio test. — In a series of real valued terms, $u_1 + u_2 + u_3 + \cdots$, suppose that

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = R. \tag{1}$$

General Ratio Test. — **A.** *If* $R < 1$, *the series converges.*

B. *If* $R > 1$, *the series diverges.*

C. *If* $R = 1$, *no conclusion can be drawn.*

Proof of A. — Notice that $\left| \frac{u_{n+1}}{u_n} \right|$ is the test ratio for the series $| u_1 | + | u_2 | + \cdots$, where all terms are positive. Since $R < 1$, this series converges, by the ratio test for a series of positive terms. Hence, by Section 232, the given series

$$u_1 + u_2 + \cdots \text{ converges.}$$

Proof of B. — Since $R > 1$, it follows from (1) that, for all values of n sufficiently large, $\left| \frac{u_{n+1}}{u_n} \right| > 1$, or $| u_{n+1} | > | u_n |$. That is, beyond a certain place in the series, each term is larger, numerically, than the preceding term, and hence the nth term does not approach zero as $n \to \infty$. Therefore, by Note 1 of Section 223, $u_1 + u_2 + \cdots$ diverges.

234. Summary of tests for convergence:

I. *If* u_n *does not approach zero, as* $n \to \infty$, *the series diverges.*

II. *The test for convergence of alternating series.*

III. *The ratio test.*

IV. *The comparison test for series of positive terms.*

NOTE. — Tests I and II should be used, when they apply conveniently. In a series of positive and negative terms, if the ratio test fails, make all terms positive and use the comparison test for convergence. If the series of numerical values converges, the original series converges, by Section 232.

EXERCISE 132

Test for convergence:

1. $1 - \frac{1}{3} + \frac{1}{5} - \cdots$.

2. $\frac{2+1}{1} + \frac{2^2+1}{2} + \frac{2^3+1}{3} + \cdots$.

3. $\frac{1}{1^2+1} - \frac{1}{2^2+1} + \frac{1}{3^2+1} - \cdots$.

4. $\frac{2}{1!} - \frac{2^2}{2!} + \frac{2^3}{3!} - \cdots$.

5. $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots$.

6. $\frac{1}{1.1} + \frac{1}{1.11} + \frac{1}{1.111} + \cdots$.

7. $\frac{1!}{5} + \frac{2!}{5^2} + \frac{3!}{5^3} + \cdots$.

8. $\frac{2}{3} \cdot \frac{1}{1} + \frac{3}{4} \cdot \frac{1}{2} + \frac{4}{5} \cdot \frac{1}{3} + \cdots$.

9. $\frac{2!}{3} - \frac{3!}{3^2} + \frac{4!}{3^3} - \cdots$.

10. $\frac{1}{2 \cdot 4} - \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} - \cdots$.

235. Power series. — If c_0, c_1, c_2, $\cdots$ are constants, then

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1} + \cdots \qquad (1)$$

is called a **power series** in x. Every power series converges if $x = 0$. A power series may converge for all values of x, or for no values of x except $x = 0$, but usually a series converges for some values of x besides $x = 0$ and diverges for other values. *The ratio test is used in finding the values for which a power series converges.*

Example 1. — For what values of x does the series converge:

$$1 - \frac{x^2}{2 \cdot 2^2} + \frac{x^4}{3 \cdot 2^4} - \cdots \qquad (1)$$

Solution. — 1. We find that $u_n = \pm \frac{x^{2n-2}}{n \cdot 2^{2n-2}}$; $u_{n+1} = \mp \frac{x^{2n}}{(n+1)2^{2n}}$.

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{x^2}{2^2} \cdot \frac{n}{n+1} \right| = \left| \frac{x^2}{4} \right| \cdot 1 = \left| \frac{x^2}{4} \right|.$$

2. By the ratio test, the series

converges if $\left| \frac{x^2}{4} \right| < 1$, or if $|x^2| < 4$, or if $|x| < 2$, and

diverges if $\left|\frac{x^2}{4}\right| > 1$, or if $|x^2| > 4$, or if $|x| > 2$.

The ratio test fails if $\left|\frac{x^2}{4}\right| = 1$, or if $|x^2| = 4$, or if $x = \pm 2$.

3. If $x = \pm 2$, the original series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ which converges, by the test for alternating series. Hence, the series converges if $-2 \leqq x \leqq 2$ and diverges if $x < -2$ or if $x > 2$. These conclusions are represented graphically in the diagram.

NOTE. — For any power series in x, it can be proved that, when the values of x for which the series converges are represented graphically, these values form an interval with zero at its center. This interval is called the **interval of convergence** of the series. A series may or may not converge for the value of x at either end point of its interval of convergence.

EXERCISE 133

Find the values of x for which the series converges, and represent these values graphically:

1. $1 + \frac{x}{2} + \frac{x^2}{2^2} + \cdots$.

2. $1 - \frac{x}{3} + \frac{x^2}{3^2} - \cdots$.

3. $1 - x + x^2 - x^3 + \cdots$.

4. $1 + \frac{x}{2} + \frac{x^2}{3} + \cdots$.

5. $1 + 10x + 10^2x^2 + \cdots$.

6. $1 - \frac{x^2}{2} + \frac{x^4}{2^2} - \cdots$.

7. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$.

8. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$.

9. $1 + 2x + 3x^2 + 4x^3 + \cdots$.

10. $1 + 3!\,x^3 + 5!\,x^5 + \cdots$.

TABLE I.—POWERS AND ROOTS

No.	Sq.	Sq. Root	Cube	Cube Root	No.	Sq.	Sq. Root	Cube	Cube Root
1	1	1.000	1	1.000	51	2,601	7.141	132,651	3.708
2	4	1.414	8	1.260	52	2,704	7.211	140,608	3.732
3	9	1.732	27	1.442	53	2,809	7.280	148,877	3.756
4	16	2.000	64	1.587	54	2,916	7.348	157,464	3.780
5	25	2.236	125	1.710	55	3,025	7.416	166,375	3.803
6	36	2.449	216	1.817	56	3,136	7.483	175,616	3.826
7	49	2.646	343	1.913	57	3,249	7.550	185,193	3.848
8	64	2.828	512	2.000	58	3,364	7.616	195,112	3.871
9	81	3.000	729	2.080	59	3,481	7.681	205,379	3.893
10	100	3.162	1,000	2.154	60	3,600	7.746	216,000	3.915
11	121	3.317	1,331	2.224	61	3,721	7.810	226,981	3.936
12	144	3.464	1,728	2.289	62	3,844	7.874	238,328	3.958
13	169	3.606	2,197	2.351	63	3,969	7.937	250,047	3.979
14	196	3.742	2,744	2.410	64	4,096	8.000	262,144	4.000
15	225	3.873	3,375	2.466	65	4,225	8.062	274,625	4.021
16	256	4.000	4,096	2.520	66	4,356	8.124	287,496	4.041
17	289	4.123	4,913	2.571	67	4,489	8.185	300,763	4.062
18	324	4.243	5,832	2.621	68	4,624	8.246	314,432	4.082
19	361	4.359	6,859	2.668	69	4,761	8.307	328,509	4.102
20	400	4.472	8,000	2.714	70	4,900	8.367	343,000	4.121
21	441	4.583	9,261	2.759	71	5,041	8.426	357,911	4.141
22	484	4.690	10,648	2.802	72	5,184	8.485	373,248	4.160
23	529	4.796	12,167	2.844	73	5,329	8.544	389,017	4.179
24	576	4.899	13,824	2.884	74	5,476	8.602	405,224	4.198
25	625	5.000	15,625	2.924	75	5,625	8.660	421,875	4.217
26	676	5.099	17,576	2.962	76	5,776	8.718	438,976	4.236
27	729	5.196	19,683	3.000	77	5,929	8.775	456,533	4.254
28	784	5.291	21,952	3.037	78	6,084	8.832	474,552	4.273
29	841	5.385	24,389	3.072	79	6,241	8.888	493,039	4.291
30	900	5.477	27,000	3.107	80	6,400	8.944	512,000	4.309
31	961	5.568	29,791	3.141	81	6,561	9.000	531,441	4.327
32	1,024	5.657	32,768	3.175	82	6,724	9.055	551,368	4.344
33	1,089	5.745	35,937	3.208	83	6,889	9.110	571,787	4.362
34	1,156	5.831	39,304	3.240	84	7,056	9.165	592,704	4.380
35	1,225	5.916	42,875	3.271	85	7,225	9.220	614,125	4.397
36	1,296	6.000	46,656	3.302	86	7,396	9.274	636,056	4.414
37	1,369	6.083	50,653	3.332	87	7,569	9.327	658,503	4.431
38	1,444	6.164	54,872	3.362	88	7,744	9.381	681,472	4.448
39	1,521	6.245	59,319	3.391	89	7,921	9.434	704,969	4.465
40	1,600	6.325	64,000	3.420	90	8,100	9.487	729,000	4.481
41	1,681	6.403	68,921	3.448	91	8,281	9.539	753,571	4.498
42	1,764	6.481	74,088	3.476	92	8,464	9.592	778,688	4.514
43	1,849	6.557	79,507	3.503	93	8,649	9.643	804,357	4.531
44	1,936	6.633	85,184	3.530	94	8,836	9.695	830,584	5.547
45	2,025	6.708	91,125	3.557	95	9,025	9.747	857,375	4.563
46	2,116	6.782	97,336	3.583	96	9,216	9.798	884,736	4.579
47	2,209	6.856	103,823	3.609	97	9,409	9.849	912,673	4.595
48	2,304	6.928	110,592	3.634	98	9,604	9.899	941,192	4.610
49	2,401	7.000	117,649	3.659	99	9,801	9.950	970,299	4.626
50	2,500	7.071	125,000	3.684	100	10,000	10.000	1,000,000	4.642

TABLE II. — TRIGONOMETRIC FUNCTIONS

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
0°	.000	1.000	.000	**45°**	.707	.707	1.000
1°	.018	.999	.018	**46°**	.719	.695	1.036
2°	.035	.999	.035	**47°**	.731	.682	1.072
3°	.052	.998	.052	**48°**	.743	.669	1.111
4°	.070	.997	.070	**49°**	.755	.656	1.150
5°	.087	.996	.087	**50°**	.766	.643	1.192
6°	.105	.994	.105	**51°**	.777	.629	1.235
7°	.122	.992	.123	**52°**	.788	.616	1.280
8°	.139	.990	.141	**53°**	.799	.602	1.327
9°	.156	.988	.158	**54°**	.809	.588	1.376
10°	.174	.985	.176	**55°**	.819	.574	1.428
11°	.191	.982	.194	**56°**	.829	.559	1.483
12°	.208	.978	.213	**57°**	.839	.545	1.540
13°	.225	.974	.231	**58°**	.848	.530	1.600
14°	.242	.970	.249	**59°**	.857	.515	1.664
15°	.259	.966	.268	**60°**	.866	.500	1.732
16°	.276	.961	.287	**61°**	.875	.485	1.804
17°	.292	.956	.306	**62°**	.883	.469	1.881
18°	.309	.951	.325	**63°**	.891	.454	1.963
19°	.326	.946	.344	**64°**	.899	.438	2.050
20°	.342	.940	.364	**65°**	.906	.423	2.144
21°	.358	.934	.384	**66°**	.914	.407	2.246
22°	.375	.927	.404	**67°**	.921	.391	2.356
23°	.391	.921	.424	**68°**	.927	.375	2.475
24°	.407	.914	.445	**69°**	.934	.358	2.605
25°	.423	.906	.466	**70°**	.940	.342	2.747
26°	.438	.899	.488	**71°**	.946	.326	2.904
27°	.454	.891	.510	**72°**	.951	.309	3.078
28°	.469	.883	.532	**73°**	.956	.292	3.271
29°	.485	.875	.554	**74°**	.961	.276	3.487
30°	.500	.866	.577	**75°**	.966	.259	3.732
31°	.515	.857	.601	**76°**	.970	.242	4.011
32°	.530	.848	.625	**77°**	.974	.225	4.331
33°	.545	.839	.649	**78°**	.978	.208	4.705
34°	.559	.829	.675	**79°**	.982	.191	5.145
35°	.574	.819	.700	**80°**	.985	.174	5.671
36°	.588	.809	.727	**81°**	.988	.156	6.314
37°	.602	.799	.754	**82°**	.990	.139	7.115
38°	.616	.788	.781	**83°**	.992	.122	8.144
39°	.629	.777	.810	**84°**	.994	.105	9.514
40°	.643	.766	.839	**85°**	.996	.087	11.430
41°	.656	.755	.869	**86°**	.997	.070	14.300
42°	.669	.743	.900	**87°**	.998	.052	19.081
43°	.682	.731	.933	**88°**	.999	.035	28.636
44°	.695	.719	.966	**89°**	.999	.018	57.290

TABLE III.—AMERICAN EXPERIENCE TABLE OF MORTALITY

Age	Number Living	Number Dying	Yearly Probability of Dying	Yearly Probability of Living	Age	Number Living	Number Dying	Yearly Probability of Dying	Yearly Probability of Living
10	100 000	749	0.007 490	0.992 510	53	66 797	1 091	0.016 333	0.983 667
11	99 251	746	0.007 516	0.992 484	54	65 706	1 143	0.017 396	0.982 604
12	98 505	743	0.007 543	0.992 457	55	64 563	1 199	0.018 571	0.981 429
13	97 762	740	0.007 569	0.992 431	56	63 364	1 260	0.019 885	0.980 115
14	97 022	737	0.007 596	0.992 404	57	62 104	1 325	0.021 335	0.978 665
15	96 285	735	0.007 634	0.992 366	58	60 779	1 394	0.022 936	0.977 064
16	95 550	732	0.007 661	0.992 339	59	59 385	1 468	0.024 720	0.975 280
17	94 818	729	0.007 688	0.992 312	60	57 917	1 546	0.026 693	0.973 307
18	94 089	727	0.007 727	0.992 273	61	56 371	1 628	0.028 880	0.971 120
19	93 362	725	0.007 765	0.992 235	62	54 743	1 713	0.031 292	0.968 708
20	92 637	723	0.007 805	0.992 195	63	53 030	1 800	0.033 943	0.966 057
21	91 914	722	0.007 855	0.992 145	64	51 230	1 889	0.036 873	0.963 127
22	91 192	721	0.007 906	0.992 094	65	49 341	1 980	0.040 129	0.959 871
23	90 471	720	0.007 958	0.992 042	66	47 361	2 070	0.043 707	0.956 293
24	89 751	719	0.008 011	0.991 989	67	45 291	2 158	0.047 647	0.952 353
25	89 032	718	0.008 065	0.991 935	68	43 133	2 243	0.052 002	0.947 998
26	88 314	718	0.008 130	0.991 870	69	40 890	2 321	0.056 762	0.943 238
27	87 596	718	0.008 197	0.991 803	70	38 569	2 391	0.061 993	0.938 007
28	86 878	718	0.008 264	0.991 736	71	36 178	2 448	0.067 665	0.932 335
29	86 160	719	0.008 345	0.991 655	72	33 730	2 487	0.073 733	0.926 267
30	85 441	720	0.008 427	0.991 573	73	31 243	2 505	0.080 178	0.919 822
31	84 721	721	0.008 510	0.991 490	74	28 738	2 501	0.087 028	0.912 972
32	84 000	723	0.008 607	0.991 393	75	26 237	2 476	0.094 371	0.905 629
33	83 277	726	0.008 718	0.991 282	76	23 761	2 431	0.102 311	0.897 689
34	82 551	729	0.008 831	0.991 169	77	21 330	2 369	0.111 064	0.888 936
35	81 822	732	0.008 946	0.991 054	78	18 961	2 291	0.120 827	0.879 173
36	81 090	737	0.009 089	0.990 911	79	16 670	2 196	0.131 734	0.868 266
37	80 353	742	0.009 234	0.990 776	80	14 474	2 091	0.144 466	0.855 534
38	79 611	749	0.009 408	0.990 592	81	12 383	1 964	0.158 605	0.841 395
39	78 862	756	0.009 586	0.990 414	82	10 419	1 816	0.174 297	0.825 703
40	78 106	765	0.009 794	0.990 206	83	8 603	1 648	0.191 561	0.808 439
41	77 341	774	0.010 008	0.989 992	84	6 955	1 470	0.211 359	0.788 641
42	76 567	785	0.010 252	0.989 748	85	5 485	1 292	0.235 552	0.764 448
43	75 782	797	0.010 517	0.989 483	86	4 193	1 114	0.265 681	0.734 319
44	74 985	812	0.010 829	0.989 171	87	3 079	933	0.303 020	0.696 980
45	74 173	828	0.011 163	0.988 837	88	2 146	744	0.346 692	0.653 308
46	73 345	848	0.011 562	0.988 438	89	1 402	555	0.395 863	0.604 137
47	72 497	870	0.012 000	0.988 000	90	847	385	0.454 545	0.545 455
48	71 627	896	0.012 509	0.987 491	91	462	246	0.532 468	0.467 534
49	70 731	927	0.013 106	0.986 894	92	216	137	0.634 259	0.365 741
50	69 804	962	0.013 781	0.986 219	93	79	58	0.734 177	0.265 823
51	68 842	1 011	0.014 541	0.985 459	94	21	18	0.857 143	0.142 857
52	67 841	1 044	0.015 389	0.984 611	95	3	3	1.000 000	0.000 000

TABLE IV.—COMPOUND AMOUNT: $(1+r)^k$

k	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	4%	5%	6%
1	1.0150	1.0200	1.0250	1.0300	1.0400	1.0500	1.0600
2	1.0302	1.0404	1.0506	1.0609	1.0816	1.1025	1.1236
3	1.0457	1.0612	1.0769	1.0927	1.1249	1.1576	1.1910
4	1.0614	1.0824	1.1038	1.1255	1.1699	1.2155	1.2625
5	1.0773	1.1041	1.1314	1.1593	1.2167	1.2763	1.3382
6	1.0934	1.1262	1.1597	1.1941	1.2653	1.3401	1.4185
7	1.1098	1.1487	1.1887	1.2299	1.3159	1.4071	1.5036
8	1.1265	1.1717	1.2184	1.2668	1.3686	1.4775	1.5938
9	1.1434	1.1951	1.2489	1.3048	1.4233	1.5513	1.6895
10	1.1605	1.2190	1.2801	1.3439	1.4802	1.6289	1.7908
11	1.1779	1.2434	1.3121	1.3842	1.5395	1.7103	1.8983
12	1.1956	1.2682	1.3449	1.4258	1.6010	1.7959	2.0122
13	1.2136	1.2936	1.3785	1.4685	1.6651	1.8856	2.1329
14	1.2318	1.3195	1.4130	1.5126	1.7317	1.9799	2.2609
15	1.2502	1.3459	1.4483	1.5580	1.8009	2.0789	2.3966
16	1.2690	1.3728	1.4845	1.6047	1.8730	2.1829	2.5404
17	1.2880	1.4002	1.5216	1.6528	1.9479	2.2920	2.6928
18	1.3073	1.4282	1.5597	1.7024	2.0258	2.4066	2.8543
19	1.3270	1.4568	1.5987	1.7535	2.1068	2.5270	3.0256
20	1.3469	1.4859	1.6386	1.8061	2.1911	2.6533	3.2071
21	1.3671	1.5157	1.6796	1.8603	2.2788	2.7860	3.3996
22	1.3876	1.5460	1.7216	1.9161	2.3699	2.9253	3.6035
23	1.4084	1.5769	1.7646	1.9736	2.4647	3.0715	3.8197
24	1.4295	1.6084	1.8087	2.0328	2.5633	3.2251	4.0489
25	1.4509	1.6406	1.8539	2.0938	2.6658	3.3864	4.2919
26	1.4727	1.6734	1.9003	2.1566	2.7725	3.5557	4.5494
27	1.4948	1.7069	1.9478	2.2213	2.8834	3.7335	4.8223
28	1.5172	1.7410	1.9965	2.2879	2.9987	3.9201	5.1117
29	1.5400	1.7758	2.0464	2.3566	3.1187	4.1161	5.4184
30	1.5631	1.8114	2.0976	2.4273	3.2434	4.3219	5.7435
31	1.5865	1.8476	2.1500	2.5001	3.3731	4.5380	6.0881
32	1.6103	1.8845	2.2038	2.5751	3.5081	4.7649	6.4534
33	1.6345	1.9222	2.2589	2.6523	3.6484	5.0032	6.8406
34	1.6590	1.9607	2.3153	2.7319	3.7943	5.2533	7.2510
35	1.6839	1.9999	2.3732	2.8139	3.9461	5.5160	7.6861
36	1.7091	2.0399	2.4325	2.8983	4.1039	5.7918	8.1473
37	1.7348	2.0807	2.4933	2.9852	4.2681	6.0814	8.6361
38	1.7608	2.1223	2.5557	3.0748	4.4388	6.3855	9.1543
39	1.7872	2.1647	2.6196	3.1670	4.6164	6.7048	9.7035
40	1.8140	2.2080	2.6851	3.2620	4.8010	7.0400	10.2857
41	1.8412	2.2522	2.7522	3.3599	4.9931	7.3920	10.9029
42	1.8688	2.2972	2.8210	3.4607	5.1928	7.7616	11.5570
43	1.8969	2.3432	2.8915	3.5645	5.4005	8.1497	12.2505
44	1.9253	2.3901	2.9638	3.6715	5.6165	8.5572	12.9855
45	1.9542	2.4379	3.0379	3.7816	5.8412	8.9850	13.7646
46	1.9835	2.4866	3.1139	3.8950	6.0748	9.4343	14.5905
47	2.0133	2.5363	3.1917	4.0119	6.3178	9.9060	15.4659
48	2.0435	2.5871	3.2715	4.1323	6.5705	10.4013	16.3939
49	2.0741	2.6388	3.3533	4.2562	6.8333	10.9213	17.3775
50	2.1052	2.6916	3.4371	4.3839	7.1067	11.4674	18.4202

V.—PRESENT VALUE OF $1 DUE AFTER k PERIODS: $(1+r)^{-k}$

k	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	4%	5%	6%
1	.985 22	.980 39	.97561	.97087	.96154	.95238	.94340
2	.970 66	.961 17	.95181	.94260	.92456	.90703	.89000
3	.956 32	.942 32	.92860	.91514	.88900	.86384	.83962
4	.942 18	.923 85	.90595	.88849	.85480	.82270	.79209
5	.928 26	.905 73	.88385	.86261	.82193	.78353	.74726
6	.914 54	.887 97	.86230	.83748	.79031	.74622	.70496
7	.901 03	.870 56	.84127	.81309	.75992	.71068	.66506
8	.887 71	.853 49	.82075	.78941	.73069	.67684	.62741
9	.874 59	.836 76	.80073	.76642	.70259	.64461	.59190
10	.861 67	.820 35	.78120	.74409	.67556	.61391	.55839
11	.848 93	.804 26	.76214	.72242	.64958	.58468	.52679
12	.836 39	.788 49	.74356	.70138	.62460	.55684	.49697
13	.824 03	.773 03	.72542	.68095	.60057	.53032	.46884
14	.811 85	.757 88	.70773	.66112	.57748	.50507	.44230
15	.799 85	.743 01	.69047	.64186	.55526	.48102	.41727
16	.788 03	.728 45	.67362	.62317	.53391	.45811	.39365
17	.776 39	.714 16	.65720	.60502	.51337	.43630	.37136
18	.764 91	.700 16	.64117	.58739	.49363	.41552	.35034
19	.753 61	.686 43	.62553	.57029	.47464	.39573	.33051
20	.742 47	.672 97	.61027	.55368	.45639	.37689	.31180
21	.731 50	.659 78	.59539	.53755	.43883	.35894	.29416
22	.720 69	.646 84	.58086	.52189	.42196	.34185	.27751
23	.710 04	.634 16	.56670	.50669	.40573	.32557	.26180
24	.699 54	.621 72	.55288	.49193	.39012	.31007	.24698
25	.689 21	.609 53	.53939	.47761	.37512	.29530	.23300
26	.679 02	.597 58	.52623	.46369	.36065	.28124	.21981
27	.668 99	.585 86	.51340	.45019	.34682	.26785	.20737
28	.659 10	.574 37	.50088	.43708	.33348	.25509	.19563
29	.649 36	.563 11	.48866	.42435	.32069	.24295	.18456
30	.639 76	.552 07	.47674	.41199	.30832	.23138	.17411
31	.630 31	.541 25	.46511	.39999	.29646	.22036	.16425
32	.620 99	.530 63	.45377	.38834	.28506	.20987	.15496
33	.611 82	.520 23	.44270	.37703	.27409	.19987	.14619
34	.602 77	.510 03	.43191	.36604	.26355	.19035	.13791
35	.593 87	.500 03	.42137	.35538	.25342	.18129	.13011
36	.585 09	.490 22	.41109	.34503	.24367	.17266	.12274
37	.576 44	.480 61	.40107	.33498	.23430	.16444	.11579
38	.567 92	.471 19	.39128	.32523	.22529	.15661	.10924
39	.559 53	.461 95	.38174	.31575	.21662	.14915	.10306
40	.551 26	.452 89	.37243	.30656	.20829	.14205	.09722
41	.543 12	.444 01	.36335	.29763	.20028	.13528	.09172
42	.535 09	.435 30	.35448	.28896	.19257	.12884	.08653
43	.527 18	.426 77	.34584	.28054	.18517	.12270	.08163
44	.519 39	.418 40	.33740	.27237	.17805	.11686	.07701
45	.511 71	.410 20	.32917	.26444	.17120	.11130	.07265
46	.504 15	.402 15	.32115	.25674	.16461	.10600	.06854
47	.496 70	.394 27	.31331	.24926	.15828	.10095	.06466
48	.489 36	.386 54	.30567	.24200	.15219	.09614	.06100
49	.482 13	.378 96	.29822	.23495	.14634	.09156	.05755
50	.475 00	.371 53	.29094	.22811	.14071	.08720	.05429

TABLE VI.—AMOUNT OF AN ANNUITY: $(s_{\overline{n}|} \text{ at } i)$

n	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	4%	5%	6%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0150	2.0200	2.0250	2.0300	2.0400	2.0500	2.0600
3	3.0452	3.0604	3.0756	3.0909	3.1216	3.1525	3.1836
4	4.0909	4.1216	4.1525	4.1836	4.2465	4.3101	4.3746
5	5.1523	5.2040	5.2563	5.3091	5.4163	5.5256	5.6371
6	6.2296	6.3081	6.3877	6.4684	6.6330	6.8019	6.9753
7	7.3230	7.4343	7.5474	7.6625	7.8983	8.1420	8.3938
8	8.4328	8.5830	8.7361	8.8923	9.2142	9.5491	9.8975
9	9.5593	9.7546	9.9545	10.1591	10.5828	11.0266	11.4913
10	10.7027	10.9497	11.2034	11.4639	12.0061	12.5779	13.1808
11	11.8633	12.1687	12.4835	12.8078	13.4864	14.2068	14.9716
12	13.0412	13.4121	13.7956	14.1920	15.0258	15.9171	16.8699
13	14.2368	14.6803	15.1404	15.6178	16.6268	17.7130	18.8821
14	15.4504	15.9739	16.5190	17.0863	18.2919	19.5986	21.0151
15	16.6821	17.2934	17.9319	18.5989	20.0236	21.5786	23.2760
16	17.9324	18.6393	19.3802	20.1569	21.8245	23.6575	25.6725
17	19.2014	20.0121	20.8647	21.7616	23.6975	25.8404	28.2129
18	20.4894	21.4123	22.3863	23.4144	25.6454	28.1324	30.9057
19	21.7967	22.8406	23.9460	25.1169	27.6712	30.5390	33.7600
20	23.1237	24.2974	25.5447	26.8704	29.7781	33.0660	36.7856
21	24.4705	25.7833	27.1833	28.6765	31.9692	35.7193	39.9927
22	25.8376	27.2990	28.8629	30.5368	34.2480	38.5052	43.3923
23	27.2251	28.8450	30.5844	32.4529	36.6179	41.4305	46.9958
24	28.6335	30.4219	32.3490	34.4265	39.0826	44.5020	50.8156
25	30.0630	32.0303	34.1578	36.4593	41.6459	47.7271	54.8645
26	31.5140	33.6709	36.0117	38.5530	44.3117	51.1135	59.1564
27	32.9867	35.3443	37.9120	40.7096	47.0842	54.6691	63.7058
28	34.4815	37.0512	39.8598	42.9309	49.9676	58.4026	68.5281
29	35.9987	38.7922	41.8563	45.2189	52 9663	62.3227	73.6398
30	37.5387	40.5681	43.9027	47.5754	56.0849	66.4388	79.0582
31	39.1018	42.3794	46.0003	50.0027	59.3283	70.7608	84.8017
32	40.6883	44.2270	48.1503	52.5028	62.7015	75.2988	90.8898
33	42.2986	46.1116	50.3540	55.0778	66.2095	80.0638	97.3432
34	43.9331	48.0338	52.6129	57.7302	69.8579	85.0670	104.1838
35	45.5921	49.9945	54.9282	60.4621	73.6522	90.3203	111.4348
36	47.2760	51.9944	57.3014	63.2759	77.5983	95.8363	119.1209
37	48.9851	54.0343	59.7339	66.1742	81.7022	101.6281	127.2681
38	50.7199	56.1149	62.2273	69.1594	85.9703	107.7095	135.9042
39	52.4807	58.2372	64.7830	72.2342	90.4091	114.0950	145.0585
40	54.2679	60.4020	67.4026	75.4013	95.0255	120.7998	154.7620
41	56.0819	62.6100	70.0876	78.6633	99.8265	127.8398	165.0477
42	57.9231	64.8622	72.8398	82.0232	104.8196	135.2318	175.9505
43	59.7920	67.1595	75.6608	85.4839	110.0124	142.9933	187.5076
44	61.6889	69.5027	78.5523	89.0484	115.4129	151.1430	199.7580
45	63.6142	71.8927	81.5161	92.7199	121.0294	159.7002	212.7435
46	65.5684	74.3306	84.5540	96.5015	126.8706	168.6852	226.5081
47	67.5519	76.8172	87.6679	100.3965	132.9454	178.1194	241.0986
48	69.5652	79.3535	90.8596	104.4084	139.2632	188.0254	256.5645
49	71.6087	81.9406	94.1311	108.5406	145.8337	198.4267	272.9584
50	73.6828	84.5794	97.4843	112.7969	152.6671	209.3480	290.3359

TABLE VII. — PRESENT VALUE OF AN ANNUITY: $(a_{\overline{n}|}\ at\ i)$

n	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	4%	5%	6%
1	.9852	.9804	.9756	.9709	.9615	.9524	.9434
2	1.9559	1.9416	1.9274	1.9135	1.8861	1.8594	1.8334
3	2.9122	2.8839	2.8560	2.8286	2.7751	2.7232	2.6730
4	3.8544	3.8077	3.7620	3.7171	3.6299	3.5460	3.4651
5	4.7826	4.7135	4.6458	4.5797	4.4518	4.3295	4.2124
6	5.6972	5.6014	5.5081	5.4172	5.2421	5.0757	4.9173
7	6.5982	6.4720	6.3494	6.2303	6.0021	5.7864	5.5824
8	7.4859	7.3255	7.1701	7.0197	6.7327	6.4632	6.2098
9	8.3605	8.1622	7.9709	7.7861	7.4353	7.1078	6.8017
10	9.2222	8.9826	8.7521	8.5302	8.1109	7.7217	7.3601
11	10.0711	9.7868	9.5142	9.2526	8.7605	8.3064	7.8869
12	10.9075	10.5753	10.2578	9.9540	9.3851	8.8633	8.3838
13	11.7315	11.3484	10.9832	10.6350	9.9856	9.3936	8.8527
14	12.5434	12.1062	11.6909	11.2961	10.5631	9.8986	9.2950
15	13.3432	12.8493	12.3814	11.9379	11.1184	10.3797	9.7122
16	14.1313	13.5777	13.0550	12.5611	11.6523	10.8378	10.1059
17	14.9076	14.2919	13.7122	13.1661	12.1657	11.2741	10.4773
18	15.6726	14.9920	14.3534	13.7535	12.6593	11.6896	10.8276
19	16.4262	15.6785	14.9789	14.3238	13.1339	12.0853	11.1581
20	17.1686	16.3514	15.5892	14.8775	13.5903	12.4622	11.4699
21	17.9001	17.0112	16.1845	15.4150	14.0292	12.8212	11.7641
22	18.6208	17.6580	16.7654	15.9369	14.4511	13.1630	12.0416
23	19.3309	18.2922	17.3321	16.4436	14.8568	13.4886	12.3034
24	20.0304	18.9139	17.8850	16.9355	15.2470	13.7986	12.5504
25	20.7196	19.5235	18.4244	17.4131	15.6221	14.0939	12.7834
26	21.3986	20.1210	18.9506	17.8768	15.9828	14.3752	13.0032
27	22.0676	20.7069	19.4640	18.3270	16.3296	14.6430	13.2105
28	22.7267	21.2813	19.9649	18.7641	16.6631	14.8981	13.4062
29	23.3761	21.8444	20.4535	19.1885	16.9837	15.1411	13.5907
30	24.0158	22.3965	20.9303	19.6004	17.2920	15.3725	13.7648
31	24.6461	22.9377	21.3954	20.0004	17.5885	15.5928	13.9291
32	25.2671	23.4683	21.8492	20.3888	17.8736	15.8027	14.0840
33	25.8790	23.9886	22.2919	20.7658	18.1476	16.0025	14.2302
34	26.4817	24.4986	22.7238	21.1318	18.4112	16.1929	14.3681
35	27.0756	24.9986	23.1452	21.4872	18.6646	16.3742	14.4982
36	27.6607	25.4888	23.5563	21.8323	18.9083	16.5469	14.6210
37	28.2371	25.9695	23.9573	22.1672	19.1426	16.7113	14.7368
38	28.8051	26.4406	24.3486	22.4925	19.3679	16.8679	14.8460
39	29.3646	26.9026	24.7303	22.8082	19.5845	17.0170	14.9491
40	29.9158	27.3555	25.1028	23.1148	19.7928	17.1591	15.0463
41	30.4590	27.7995	25.4661	23.4124	19.9931	17.2944	15.1380
42	30.9941	28.2348	25.8206	23.7014	20.1856	17.4232	15.2245
43	31.5212	28.6616	26.1664	23.9819	20.3708	17.5459	15.3062
44	32.0406	29.0800	26.5038	24.2543	20.5488	17.6628	15.3832
45	32.5523	29.4902	26.8330	24.5187	20.7200	17.7741	15.4558
46	33.0565	29.8923	27.1542	24.7754	20.8847	17.8801	15.5244
47	33.5532	30.2866	27.4675	25.0247	21.0429	17.9810	15.5890
48	34.0426	30.6731	27.7732	25.2667	21.1951	18.0772	15.6500
49	34.5247	31.0521	28.0714	25.5017	21.3415	18.1687	15.7076
50	34.9997	31.4236	28.3623	25.7298	21.4822	18.2559	15.7619

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ANSWERS TO EXERCISES

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1.	Sum = 5.	Diff. = 1.	Prod. = 6.	Quo. = $\frac{3}{2}$.
3.	Sum = 8.	Diff. = 16.	Prod. = − 48.	Quo. = − 3.
5.	Sum = − 20.	Diff. = − 12.	Prod. = 64.	Quo. = 4.
7.	Sum = − 12.	Diff. = 24.	Prod. = − 108.	Quo. = $-\frac{1}{3}$.
9.	Sum = − 7.	Diff. = − 3.	Prod. = 10.	Quo. = $\frac{5}{2}$.

Exercise 2. Page 4

1. $7a - 1$.
3. $8.55x - 4a - .4$.
5. $9z + 1$.
7. $h - 5$.
9. $6.6y + 2.4$.
11. $2x + y - 6$.
13. Sum $= x - y + z$. Difference $= 7x - 5y + 9z$.
15. 7; $2x^2 - 7x + 15$.
17. 6; $-3x^3 + 3x^2 + 6x + 5$.

Exercise 3. Page 6

1. $20x^5$.
3. $5x - 3xy$.
5. $6x^4y$.
7. $4x^2 - 5x^3y$.
9. $-15x^2y^3z$.
11. $-10x^2 + 14xy - 2xz$.
13. $-6x^2y - 10x - 2x^3$.
15. $x^3 - 10x^2 + 26x - 15$.
17. $p^4 + r^2p^2 + r^4$.
19. $b^4 + 2ab^3 - 28a^2b^2 + 26a^3b - 5a^4$.
21. $2x^3 - 11x^2 + 22x - 21$.
23. $-5x^3 - 20x^2y + 28xy^2 + 15y^3$.
25. $x^2 + (a + b)x + ab$.
27. $27a^3 + b^3$.
29. $a^3 - 2a^2b + 2ab^2 - b^3$.
31. $-14x^2 + 34xy - 12y^2 - 11x - 11y + 15$.
33. $-2x^2 - 3xy + 3y^2$.
35. $8a^2 - 3a + 3b$.

Exercise 4. Page 9

1. $-5b$.
3. $-3b$.
5. $-x^2 + x + b$.
7. $2b^4 - 3y^4$.
9. $-3a^2 + 6b$.
11. $a + 2$.

13. *Quot.* $= m + 9$; *remainder* $= 1$. **15.** $n^2 - 8$.
17. *Quotient* $= x^2 + 9y$; *remainder* $= -y^2$.
19. $x^2 - 2xy - y^2$. **21.** $m^2 - 2mn + 3n^2$.
23. *Quotient* $= n + 3$; *remainder* $= 5$.
25. $4a^2 + 6a + 9$. **27.** $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.
29. *Quotient* $= z^2 - 2z + 4$; *remainder* $= -16$.

Exercise 5. Page 11

1. $24c - 30ac$.
3. $u^2 - 2uv + v^2$.
5. $2x^2 - 4x - 6$.
7. $16x^2 + 40xy + 25y^2$.
9. $42a^2 + ab - b^2$.
11. $x^2 - \frac{9}{25}$.
13. $2 - 3a^2 - 35a^4$.
15. 9506.
17. $x^4 - 4y^4$.
19. $24m^2 + 17mn - 20n^2$.
21. $4 - 25w^4$.
23. $x^2 + 2cx - 80c^2$.
25. $6x^2 - 11xy - 35y^2$.
27. $4x^2 - x + \frac{1}{16}$.
29. $121c^4 - 64d^2$.
31. $x^4y^2 + 2ax^2y - 15a^2$.
33. $x^2 + 2xy + y^2$.
35. $x^4 - 16y^6$.
37. $21z^2 + 17x^2yz + 2x^4y^2$.
39. $4x^2 + 2xy - 12y^2$.
43. $h^2 + 2hw + w^2 - z^2$.
45. $h^2 - 2hk + 2hw + k^2 - 2kw + w^2$.
47. $a^2 + 2ab + b^2 + 2a + 2b - 35$.
49. $x^2 - 2xy + y^2 - z^2$.
51. $a^2 + 2ab + b^2 - c^2 - 2cd - d^2$.
53. $x^2 - 2xy + y^2 + 2x - 2y - 48$.
55. $x^2 + 6xy + 9y^2 + 5xz + 15yz - 150z^2$.
57. $x^4 + 4x^3 - 8x - 5$.
59. $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$.
61. $a^2 + b^2 + y^2 + 2ab + 2ay + 2by$.
63. $9z^2 + 25 + x^2 - 6xz - 30z - 10x$.

Exercise 6. Page 14

1. $(y - 7x)(y + 7x)$. **3.** $(7x - 25z)(7x + 25z)$.
5. $(2a^2 - 5b)(2a^2 + 5b)$.
7. $(w^4 + z^4)(w^2 + z^2)(w - z)(w + z)$.

9. $(8y - 6ab)(8y + 6ab)$.
11. $(4 + y)^2$.
13. $(w + 5)^2$.
15. $(a - \frac{1}{2})^2$.
17. $(m + 4n)^2$.
19. $(1 + 2xy)^2$.
21. $(t - 7)(t - 3)$.
23. $(w + 10)(w - 6)$.
25. $(2x - 9)(x - 3)$.
27. $(7r + 11s)(r - s)$.
29. $(2 - 3y)(1 - 2y)$.
31. $(x^2 + 25)(x^2 + 2)$.
33. $(5c - d)^2$.
35. $(a + 12b)(a - 5b)$.
37. $(4x^2 + 9y^2)(2x - 3y)(2x + 3y)$.
39. $r(2w + 5)(w - 4)$.
41. $(a + x - 2)^2$.
43. $(\frac{1}{2} - 6ab)(\frac{1}{2} + 6ab)$.
45. $(3mn - 5r^2)^2$.
47. $(13 + ab)(4 - ab)$.
49. $(3c + 135)(c - 1)$.
51. $(a + b + 10)(a + b - 2)$.
53. $(c - a - b)(c + a + b)$.
55. $(24z + a + b)(4z - a - b)$.
57. $(m - n - x - y)(m - n + x + y)$.
59. $(9 - x - y)(4 + x + y)$.
61. $(4x^2 + 25y^2)(2x - 5y)(2x + 5y)$.
63. $(y - x - 2)(y + x - 6)$.
65. $(1 + n - 2m)^2$.
67. $(x + 3)(x - 2)(x + 4)(x - 3)$.
69. $(16 + m + n)(9 - m - n)$.

Exercise 7. Page 16

1. $(4h - k)(x + y)$.
3. $(5c - 2d)(r - s)$.
5. $(a + 5)(y + z)$.
7. $(x - y)(a - b)$.
9. $(x - 2)(3x^2 + 1)$.
11. $(a + b - c)(a + b + c)$.
13. $(y - x - 3)(y + x + 3)$.
15. $(2c + 1 - b)(2c + 1 + b)$.
17. $(m + n)(m - n)^2$.
19. $(x + 2y - z)(x + 2y + z)$.
21. $(3y + 1)(y^2 - 2)$.
23. $(2 - c)(cd - 3)$.
25. $(x + 2 - y)(x + 2 + y)$.
27. $(s - 2t)(2s^2 + t^2)$.
29. $(5x^2 + 6)(8x - 7)$.
31. $(r + 3s - t - 5)(r - 3s + t - 5)$.
33. $(a + 3b - x + 2)(a + 3b + x + 2)$.
35. $(5a - b - c + d)(5a - b + c - d)$.

Exercise 8. Page 17

1. $(a^2 + ab + b^2)(a^2 - ab + b^2)$.
3. $(x^2 + 2x + 4)(x^2 - 2x + 4)$.

5. $(2a^2 + 2a + 3)(2a^2 - 2a + 3)$.
7. $(4a^2 + 2a - 1)(4a^2 - 2a - 1)$.
9. $(3r^2 + rt - 7t^2)(3r^2 - rt - 7t^2)$.
11. $(5z^2 + 3wz - w^2)(5z^2 - 3wz - w^2)$.
13. $(x^2 - 2x + 2)(x^2 + 2x + 2)$.
15. $(4x^2 + 6xy + 5y^2)(4x^2 - 6xy + 5y^2)$.
17. $(3x^2 + 2x - 1)(3x^2 - 2x - 1)$.
19. $(2a^2 - 2a + 1)(2a^2 + 2a + 1)$.

Exercise 9. Page 18

1. $a^3 - x^3$.
3. $(x + y)^3 + z^3$.
5. $(x - 2)(x^2 + 2x + 4)$.
7. $(3y - 2)(9y^2 + 6y + 4)$.
9. $(y + 2a)(y^2 - 2ay + 4a^2)$.
11. $(x - 1)(x^2 + x + 1)$.
13. $(a + \frac{1}{2})(a^2 - \frac{1}{2}a + \frac{1}{4})$.
15. $(\frac{1}{2}x - \frac{1}{3}y)(\frac{1}{4}x^2 + \frac{1}{6}xy + \frac{1}{9}y^2)$.

Miscellaneous Problems on Factoring. Page 19

1. $a(x - 7)(x + 5)$.
3. $(x + 2y - z)(x + 2y + z)$.
5. $(x - c)(x + c)(ax + b)$.
7. $(2c + 3a)(5c + 2a)$.
9. $(8x + 7)(5x^2 - 6)$
11. $(w - 2)(w + 1)(w^2 + 2w + 4)(w^2 - w + 1)$.
13. $(6 - a - 6b)(6 + a + 6b)$.
15. $(2 + 3w)(4 - 6w + 9w^2)$.
17. $(x - y + 3)(x + y + 3)$.
19. $(3z + 1)(z + 7)$.
21. $(3c - 5d)(c + d)(c - d)(3c + 5d)$.
23. $(a + b - 2x)(a + b + 3x)$.
25. $y^2(6x - 5)(3x + 2)$.
27. $(c^2 - 2cd + 2d^2)(c^2 + 2cd + 2d^2)$.
29. $y^2(2a - 7)^2$.
31. $(3m^2 + 2mt + t^2)(3m^2 - 2mt + t^2)$.
33. $(2cd + 3x)(2cd - 7x)$.
35. $(.5x - .3y)^2$.
37. $(3y + z)(5a - 2b)$.
39. $(2z^2 - 3)(2z - 5)$.
41. $(y - 5 - 3x + z)(y - 5 + 3x - z)$.
43. $(2r - s)(x - 3y + z)$.
45. $(2x - y)(x^2 + x + 1)$.
47. $a^3 + 3a^2b + 3ab^2 + b^3$.
49. $27 + 27a + 9a^2 + a^3$.
51. $27x^3 + 54x^2 + 36x + 8$.
53. $8 - 6x + \frac{3}{2}x^2 - \frac{1}{8}x^3$.

Exercise 10. Page 21

1. $\frac{28}{5}$.

3. $\frac{9x^2 - y^2}{x + y}$.

5. $\frac{3y - 3x}{c + d}$.

7. $\frac{5}{12}$.

9. $\frac{3}{7}$.

11. $\frac{5x^2}{3y}$.

13. $\frac{4}{5x}$.

15. $3ab$.

17. $\frac{3 + 4d}{4 + 5d}$.

19. $\frac{m + 8}{m + 6}$.

21. $\frac{1}{a + 2b}$.

23. $\frac{a(m - n)}{m + n}$.

25. $\frac{b(a - 2)}{c(a - 6)}$.

27. $\frac{a^2 + ab + b^2}{a + b}$.

29. $\frac{y + 10z}{y + 4z}$.

31. $-\frac{3x + 7y}{4y}$.

33. $\frac{a + 4b}{a + 7b}$.

35. $-\frac{2a + 3b}{3a + 2b}$.

Exercise 11. Page 22

1. $-\frac{2}{3}$.

3. $-\frac{1}{4}$.

5. $-\frac{3d}{c^2}$.

7. -1.

9. $-\frac{3x}{y^2}$.

11. $\frac{d - c}{c}$.

13. $-\frac{r^2 + 3}{2(r^2 + 2)}$.

15. $-\frac{t + 2}{t + 4}$.

Exercise 12. Page 24

1. 30.

3. $42a^2b$.

5. $72y^3z^2$.

7. $12(x - 3)$.

9. $2(y - 5)(y + 5)$.

11. $(x - 2)^3(x + 3)$.

13. $(y - 5)^2(y + 5)$.

15. $(x + 6)(x - 4)(x - 7)(x + 7)$.

17. $6(x - y)(x^2 + xy + y^2)(x + 3y)(x + y)$.

19. $2a(a + 5b)^2(2a + b)$.

21. $(y - 2)(y - 3)(y + 4)$.

Exercise 13. Page 26

1. $\frac{4x - 83}{35}$.

3. $\frac{27a^2 - 15ab - 20a + 24b}{12a^2b}$.

5. $\frac{3a^2 - 20a^2c - 24c^2 + 4ac^2}{4ac}$.

7. $\frac{1 - 10x - 4x^2}{(x - 2)(x + 3)}$.

9. $\frac{4a^2 - 4a + 1}{(a - 1)(a + 1)}$.

11. $\frac{5n}{6(a - n)}$.

13. $\frac{3a^2 - 23a - 2}{3a}$.

15. $\frac{2m - am - 4a}{m(m - 4)(m + 4)}$.

17. $\frac{12}{(a + 10)(a - 6)}$.

19. $\frac{x^2 - yx^2 - xy}{(x - y)(x^2 + xy + y^2)}$.

21. $\frac{x^2 - 11x + 29}{x(x-5)}$.

23. $\frac{r^2 + 5ar - 4a^2}{(r-3a)^2(r+7a)}$.

25. $\frac{2a^2 + 18a + 3}{(3a+1)(2a-1)(a+2)}$.

Exercise 14. Page 28

1. $\frac{6s}{3}$.

3. $\frac{3}{40}$.

5. $\frac{7ac^2}{15}$.

7. $\frac{15a^3}{56b^3}$.

9. $\frac{15b}{2a}$.

11. $\frac{3(y+3)}{2t^2(y-2)}$.

13. $\frac{r(r+2s)}{s}$.

15. $\frac{3m(2m+3)}{(m-1)(2m-1)}$.

17. $\frac{a+b}{a(a+3b)}$.

19. $\frac{(2a-3b)(3a-5b)}{(3a+5b)(3a-2b)}$.

Exercise 15. Page 30

1. $\frac{5}{3}$.

3. $\frac{51}{8}$.

5. $\frac{2}{3x-1}$.

7. $\frac{1}{2(2y+1)}$.

9. $\frac{6b}{10x - x^2}$.

11. $\frac{xy(5x+3y)}{3(y^2 - 2x^2)}$.

13. $\frac{x+2d}{x-d}$.

15. $\frac{(x-y)^2(x+2y)}{y}$.

17. $\frac{a^2 - ab + b^2}{a^2 + ab + b^2}$.

19. $\frac{2a^2 + 7a - 1}{7}$.

21. $\frac{2ab(a-3b)}{3}$.

23. $\frac{3x+4}{x+1}$.

25. $\frac{1}{75}$.

27. $\frac{3}{8}$.

29. $\frac{1}{27b}$.

31. $\frac{3d}{5c}$.

33. $\frac{3x+3}{4x-7}$.

35. $\frac{2a^2(7-a)}{7a^2 + 24a + 2}$.

Exercise 16. Page 35

1. $x = 2$.

3. $x = 6$.

5. $x = \frac{3}{2}$.

7. $z = -\frac{13}{8}$.

9. $x = -5$.

11. $x = \frac{1}{2}$.

13. $z = \frac{3}{2}$.

15. $x = \frac{5}{3}$.

17. $x = .3$.

19. $x = -2$.

21. $x = \frac{3+a}{b}$.

23. $x = \frac{c+d}{a-b}$.

25. $x = \frac{k}{p}$.

27. $s = \frac{5}{a+b}$.

29. $x = \dfrac{4}{d-5}$. **35.** $x = a + 1$. **41.** $t = \dfrac{s}{v}$.

31. $x = \dfrac{c + 4d}{5}$. **37.** $s = \dfrac{2m}{b}$. **43.** $m = \dfrac{2K}{v^2}$.

33. $x = \dfrac{2}{3 + 4b}$. **39.** $x = \dfrac{1}{c + d}$.

45. $P = \dfrac{A}{1 + rt}$; $r = \dfrac{A - P}{Pt}$; $t = \dfrac{A - P}{Pr}$.

Exercise 17. Page 37

1. $x = -\frac{3}{14}$. **7.** $t = \frac{2}{5}$. **13.** $x = -1$. **19.** $x = \frac{5}{2}$.

3. $a = -2$. **9.** $s = 2$. **15.** $x = 1$. **21.** $x = c - b$.

5. $x = -11$. **11.** $t = -1$. **17.** $x = 7$. **23.** $x = -2n$.

25. $x = b$. **27.** $x = 3a$. **29.** $x = -\frac{3}{4}$.

31. $a = \dfrac{bC}{Kb + C}$; $b = \dfrac{aC}{C - Ka}$. **33.** $a = \dfrac{w(600\,d^2 + l^2)}{3000\,d^2}$.

Exercise 18. Page 39

1. Base = 105; Altitude = 60. **3.** A's age is 12; B's age is 3.

5. 5 pennies; 15 nickels; 12 quarters.

7. 40 pounds of 90¢, and 60 pounds of 40¢ tea.

9. 1 gal. **13.** $3\frac{1}{13}$ hrs. **17.** 8.

11. 200 pounds. **15.** 4 days. **19.** 9.

21. (*a*) 30 mi. per hr.; (*b*) $\dfrac{300}{h}$ mi. per hr.

23. At the end of $6\frac{2}{3}$ hrs.

25. A's rate is 30 miles per hr. and B's rate is 18 miles per hr.

27. $16\frac{4}{11}$ min. after 3 o'clock. **31.** $6\frac{3}{5}$ miles per hr.

29. 5 hrs. after express starts. **33.** .04; .035; .0225.

35. (*a*) \$22.50; (*b*) \$26.25; (*c*) \$18.75; (*d*) \$11.25.

37. \$500 at 6%; \$1500 at 7%. **39.** \$13,500. **41.** 12 ft.

43. $2\frac{3}{8}$ ft. from fulcrum on side with boy weighing 70 lbs.

45. 1050 lbs.

Review Exercise 19. Page 45

1. $8x^2 - 3x + 3y$.

3. $144z^2 - 81w^2$.

5. $9z^2 + 6hz + h^2$.

7. $6x^2 + 2x - 20$.

9. $z^3 - 6z^2 + 12z - 8$.

11. $-a(2x + 6y - 9z)$.

13. $(2z - 3)(4z - 5)$.

15. $(5x - 1)^2$.

17. $(2a - 5)(2a^2 - 3)$.

19. $(x^2 - 2xy + 3y^2)(x^2 + 2xy + 3y^2)$.

21. $(5z - w)(4z + w)$.

23. $(m - n - 3)(m - n - 5)$.

25. $(2a - 6w + 1)(2a + 6w + 1)$.

27. $(7r^2 - 1 - x)(7r^2 + 1 + x)$.

29. $(b - c + 3)(b + c + 3)$.

31. $\frac{15x - 40}{2}$.

33. $\frac{20}{3x - 2}$.

35. $\frac{3m^2 + 4m - 4}{m(3m + 1)}$.

37. $\frac{2x^2 + 8x + 14}{(1 - x)(1 + x)(2 + x)}$.

39. $\frac{3y - 5x}{3y + 4x}$.

41. $3ab - b^2$.

43. $\frac{9 + 4a}{4 - a}$.

45. $P = 300$.

47. $x = -1$.

49. $y = 6$.

51. $x = \frac{1}{3}$.

53. $x = 2$.

55. $x = 3a$.

57. $x = \frac{pNn}{Mm + Nn}$.

59. $5\frac{5}{8}$ lbs.

61. $12\frac{4}{5}$ hrs.

63. $\frac{5}{10}$.

65. Each is \$5000.

67. 5 ft.

Exercise 21. Page 54

1. (*a*) $(x = 2, y = \frac{1}{5})$. (*b*) $(x = -\frac{2}{3}, y = 4)$. (*c*) $(x = 6, y = 0)$.

13. (*a*) Where $x = 7$. (*b*) Where $y = -7$.

Exercise 22. Page 56

1. $x = 2, y = -1$.

3. $u = 5, y = 1$.

5. No solution.

7. $x = 2, y = -2$.

9. $x = \frac{7}{3}, y = \frac{1}{3}$.

11. $x = -4, y = -6$.

Exercise 23. Page 58

1. $(x = 4, y = 1)$.

3. $(x = 5, y = 1)$.

5. $(y = -3, z = -1)$.

7. $(x = -1, y = 3)$.

9. $(m = -\frac{1}{4}, n = 5)$.

11. $(x = .25, y = .12)$.

13. $(x = 3, y = 1)$.

15. $(x = 3, y = -7)$.

17. $(x = \frac{1}{3}, y = 1)$.

19. $(x = 2, y = 3)$.

21. $(x = -1, y = \frac{1}{4})$.

23. $(w = 5, z = -3)$.

25. $(y = 0, z = -\frac{2}{5})$.

Exercise 24. Page 60

1. $x = \frac{1}{4a}$; $y = \frac{1}{4}$. **3.** $w = \frac{c(a+b)}{a^2+b^2}$; $z = \frac{c(b-a)}{a^2+b^2}$.

5. $x = \frac{5a^2c + 3abc}{a - 3b}$; $y = \frac{9ac}{a - 3b}$.

7. $x = \frac{d^2 + cd - d + c}{d(c+d)}$; $y = \frac{c^2 + cd - c + d}{c(c+d)}$.

9. $w = \frac{2}{3}a$; $z = -\frac{b}{3}$.

Exercise 25. Page 61

1. $x = 1$; $y = 2$; $z = -2$. **5.** $x = 2$; $y = -1$; $z = -3$.

3. $s = -1$; $t = -1$; $y = -\frac{1}{2}$. **7.** $x = \frac{7}{3}$; $y = 1$; $z = 5$.

9. $x = 1$, $y = \frac{1}{2}$, $z = \frac{1}{3}$.

Exercise 26. Page 62

1. -14. **3.** -46. **5.** $-c - 3d$. **7.** $hg - kl$.

Exercise 27. Page 64

Problems 1 to 18. — *See Problems 1 to 18, Exercise 23*

19. $x = \frac{a-b}{a^2+b^2}$; $y = \frac{a+b}{a^2+b^2}$. **21.** $x = 2d$; $y = c$.

23. $x = \frac{a+b}{ac}$; $y = \frac{a+b}{b}$.

Exercise 28. Page 65

1. -7. **3.** 18.

5. $k_1b_2c_3 + k_3b_1c_2 + k_2b_3c_1 - k_3b_2c_1 - k_1b_3c_2 - k_2b_1c_3$.

Exercise 29. Page 67. *See Exercise 25*

Miscellaneous Exercise 30. Page 67

1. $\frac{20}{3}$ and $-\frac{4}{3}$. **7.** 72.

3. 7 and $\frac{3}{2}$. **9.** 85′ long; 64′ wide.

5. length 17′; width 10′. **11.** 42.

13. Current: $2\frac{1}{2}$ mi. per hr.; crew: 8 mi. per hr.

15. 475. **17.** \$2300 at 4%; \$1700 at 7%.

19. 24 days for A; 12 days for B.

21. \$2000 at 6%; \$3000 at 4%; \$5000 at 5%.

Exercise 31. Page 71

1. $2^6 = 64$. $(-2)^5 = -32$. $(-1)^6 = 1$. $(-1)^3 = -1$. $(-3)^5 = -243$. $(-4)^4 = 256$. $(-5)^4 = 625$. $(.1)^5 = .00001$.

3. $(\frac{1}{2})^4 = \frac{1}{16}$. $(\frac{1}{3})^5 = \frac{1}{243}$. $(\frac{1}{5})^4 = \frac{1}{625}$. $(\frac{1}{6})^3 = \frac{1}{216}$. $(\frac{1}{4})^3 = \frac{1}{64}$.

5. a^{10}.

7. c^{n+h}.

9. x^{3+m}.

11. c^8.

13. x^n.

15. y^{15}.

17. $32\,a^5$.

19. a^{4n}.

21. $-.008\,x^3$.

23. a^5x^5.

25. x^3y^3.

27. $-.343\,x^3$.

29. $64\,x^3y^3$.

31. $-x^3y^6$.

33. $.00001\,a^5b^5$.

35. $5\,x^6$.

37. $r^{3n}s^n$.

39. $x^{at}y^{3a}$.

41. $256\,x^8y^4$.

43. $x^{11}y^4$.

45. $\frac{1}{y^2}$.

47. x^{n+1}.

49. $\frac{1}{z^n}$.

51. $a^{15}x^{11}$.

53. w^{2r-1}.

55. $\frac{x^6}{b^3}$.

57. $-\frac{x^3}{27\,y^3}$.

59. $-\frac{64\,x^3}{y^3z^3}$.

61. $\frac{8\,x^3}{27\,b^3}$.

63. $\frac{c^8d^4}{x^4}$.

65. $\frac{.144\,x^2}{a^2b^2}$.

67. $\frac{8\,x^{3n}}{y^{12}}$.

69. $\frac{a^{4n}}{b^{6x}}$.

Exercise 32. Page 74

1. ± 12; ± 11; ± 14; $\pm \frac{1}{2}$; $\pm .1$; ± 10; ± 1.

3. -3; 4; -5; 10; $-.1$; $.2$.

5. 2; -2; 3; -3; 10.

7. 6. **9.** 2. **11.** 2. **13.** -3. **15.** .02. **17.** $\frac{1}{2}$. **19.** 30. **21.** 7. **23.** -6.

Exercise 33. Page 77

1. 2.

3. $\frac{1}{5}$.

5. $-\frac{1}{125}$.

7. -2.

9. $\frac{1}{2}$.

11. -1.

13. $\frac{1}{4}$.

15. $\frac{1}{2}$.

17. .1.

19. 10.

21. $\frac{1}{a^2}$.

23. $\frac{x^2}{a^5z^3}$.

25. $\frac{5}{a^3}$.

27. $\frac{x^3}{25}$.

29. $\frac{5\,y}{x^3z^3}$.

31. $\frac{5\,y^5}{3\,x^4}$.

33. $\frac{1}{70{,}000\,x^2}$.

35. $\frac{5\,x^4z^2}{y^2}$.

37. $5\,a^{6}y^{-3}$.

39. $16\,h^{-2}x^{3}y^{2}z^{-3}$.

41. $a^{-1}b^{-1}$.

43. $8\,x^{\frac{1}{2}}y^{-\frac{1}{2}}z^{-1}$.

45. $\sqrt[4]{a^3}$.

47. $5\sqrt{a}$.

49. $\sqrt{3\,xy}$.

51. $\sqrt[3]{4\,a^2b^2}$.

53. $a^{\frac{2}{3}}$.

55. $x^{\frac{3}{5}}$.

57. $A^{\frac{1}{7}}$.

59. $(a+b)^{\frac{1}{4}}$.

61. $(a+x)^{\frac{2}{3}}$.

63. $x^{\frac{7}{3}}$.

65. $\dfrac{1}{a^{\frac{1}{3}}}$.

67. $xy^{\frac{2}{3}}$.

69. 25.

71. $\dfrac{b^2}{a}$.

73. $2\,a^{4b}$.

75. $\dfrac{5}{x}$.

77. $\dfrac{x^2}{25}$.

79. $\dfrac{2}{x^2}$.

81. $a^2x^{\frac{2}{3}}$.

83. $x^{\frac{7}{2}}$.

85. $a^{\frac{5}{2}}$.

87. $a^{\frac{2}{3}}x^{\frac{3}{4}}$.

89. $\dfrac{3\,x^{\frac{1}{2}}}{a}$.

91. $4\,x^{\frac{1}{6}}y^{\frac{1}{2}}$.

93. $\dfrac{4\,x^4}{y}$.

95. 25.

97. 1296.

99. $\dfrac{9\,m^2}{z^4}$.

101. $\dfrac{m^{3z}}{11\,x^{\frac{2}{3}}}$.

103. $\dfrac{3^{\frac{1}{2}}a^{\frac{1}{4}}b^{\frac{1}{6}}}{4\,c}$.

105. $\dfrac{3\,ax^2}{2}$.

107. $\dfrac{.25}{4\,x^2z^4}$.

109. $\dfrac{6\,x^3y^2}{x^2+y^2}$.

111. $\dfrac{62}{375}$.

113. $1-4\,y^2$.

115. $3\,x^3-15\,x^4-12\,x^5$.

117. $a+a^{\frac{2}{3}}x^3$.

119. $e^{2x}-2+\dfrac{1}{e^{2x}}$.

121. $\dfrac{1}{x^2}+\dfrac{2}{xy}+\dfrac{1}{y^2}$.

123. $x^{\frac{2}{3}}-y^{\frac{2}{3}}$.

125. $a-3\,ab^{\frac{1}{3}}+a^{\frac{1}{3}}b^{\frac{2}{3}}+3\,a^{\frac{1}{3}}b$.

127. $x+y$.

129. $9\,a-4\,a^{\frac{2}{3}}$.

131. $x^{\frac{2}{3}}+2\,x^{\frac{1}{2}}+x^{\frac{1}{3}}-1$.

133. $\dfrac{27}{x}-1$.

135. $(x^{-1}+y^{-1})(x^{-1}-y^{-1})$.

137. $(y^2-3\,x^{\frac{4}{3}})(y^2+3\,x^{\frac{4}{3}})$.

139. $(x^{\frac{1}{5}}-3\,y^{\frac{1}{5}})(x^{\frac{1}{5}}+3\,y^{\frac{1}{5}})$.

141. $(x^{\frac{1}{4}}+1)^2$.

143. $(a+b^{-1})^2$.

145. $(3-x^{\frac{1}{3}})(1-x^{\frac{1}{3}})$.

147. $(x^{-2}-2\,y^{-1})$.

149. (a^x-c^y).

151. $a+a^{\frac{1}{2}}b^{\frac{1}{2}}+b$.

153. $3\,y^{\frac{2}{3}}-4\,x^{\frac{1}{3}}y^{\frac{1}{3}}$.

155. $2\,cd-3\,d^2-5\,c^{-1}d^3$.

157. $x-3\,x^{\frac{1}{2}}-2$.

159. $m^4-3\,m^2+4$.

161. $y^{-1}+2$.

163. 27.

165. 8.

167. .01.

169. $\frac{1}{25}$.

Exercise 34. Page 84

1. 48.

3. -1.

5. 10.

7. $\frac{1}{2}$.

9. $-\frac{1}{4}$.

11. $5\sqrt{7}$, or 13.230.

13. $6\sqrt{3}$, or 10.392.

15. $50\sqrt{2}$, or 70.7.
17. $-8\sqrt[3]{5}$, or -13.680.
19. $18\sqrt[3]{2}$, or 22.680.
21. $15\sqrt{2}$, or 21.21.
23. 2.
25. $-3z^2$.
27. $\frac{1}{2}z^2w$.
29. $-2w^3$.
31. $\frac{1}{2}a$.
33. $2y^k$.
35. $-xy^2$.
37. $\frac{a^2}{2y^2}$.
39. $a^{2n}y^n$.
41. $-\frac{1}{3x}$.
43. $12m\sqrt[4]{2}$.
45. $-4t^2\sqrt[3]{2}$.
47. $2x^2y^n$.
49. $20m^2\sqrt[3]{3m}$.
51. $3\sqrt[5]{a^2z^3}$.
53. $.2a\sqrt[3]{a}$.
55. $-2x^3$.
57. $.4c^2d\sqrt{5d}$.
59. $2\sqrt{1+x}$.
61. $a\sqrt{b-c}$.
63. $d\sqrt[4]{d^4+3c^2}$.
65. $\frac{2x}{y^2}\sqrt{x}$.
67. $\frac{x^k}{2y^2}\sqrt{5}$.
69. $\frac{-2x^n}{3}$.
71. $\frac{2x}{y^k}\sqrt[4]{2}$.
73. $2\sqrt{7}$, or 5.292.
75. $2\sqrt{2}$, or 2.828.
77. $4\sqrt{m^2-2m}$.

Exercise 35. Page 86

1. $\frac{\sqrt{2}}{2}$, or .707.
3. $\frac{2\sqrt{3}}{3}$, or 1.155.
5. $\frac{\sqrt{30}}{18}$, or .304.
7. $\frac{\sqrt{7}}{7}$, or .378.
9. $4\sqrt[3]{2}$, or 5.040.
11. $\frac{\sqrt[3]{30}}{10}$, or .3107.
13. $\frac{\sqrt{30}}{10}$, or .5477.
15. $\frac{3}{5}\sqrt[3]{2}$, or .504.
17. $\frac{\sqrt{70}}{10}$, or .8367.
19. $\frac{\sqrt{35x}}{7x}$.
21. $\frac{y\sqrt{5x}}{10}$.
23. $\frac{\sqrt{30bx}}{6b}$.
25. $\frac{\sqrt[4]{cd^3x}}{d}$.
27. $\frac{\sqrt[3]{45a}}{t}$.
29. $-\frac{\sqrt[3]{10d^2}}{2d}$.
31. $\frac{\sqrt{5y}}{5xy}$.
33. $\frac{w\sqrt[3]{49u^2w^2}}{7u}$.
35. $\frac{\sqrt{12z-18z^3}}{6z}$.
37. $\frac{\sqrt{3abc-3abd}}{c-d}$.

Exercise 36. Page 87

1. $\sqrt[3]{5}$, or 1.710.
3. $\sqrt{2}$, or 1.414.
5. $\sqrt{7}$, or 2.646.
7. $\sqrt{5x}$.
9. $\sqrt[4]{4m}$.
11. $\sqrt{3ab}$.
13. $\frac{1}{10}\sqrt{10x}$.
15. $\sqrt{2xy}$.
17. $\frac{1}{10}\sqrt{20y}$.
19. $\sqrt[6]{14x^4y^4}$.
21. $\sqrt{2ab}$.
23. $y\sqrt{2ay}$.
25. $\frac{1}{5}\sqrt[3]{50}$.
27. $7\sqrt[3]{4x}$.
29. $\frac{5}{x}\sqrt{3}$.
31. $\frac{xy}{x-y}\sqrt{xy-y^2}$.

33. $\frac{2}{x}\sqrt{x^2+a}$. **37.** $\frac{\sqrt{a^2-4}}{a-2}$. **41.** $\frac{\sqrt{9b^2-a^2}}{3ab}$.

35. $\frac{\sqrt{a^2b+ab^2}}{ab}$. **39.** $\frac{\sqrt{125c^5(a+b)^3}}{5c}$. **43.** $\frac{c-4x}{x}$.

Exercise 37. Page 88

1. $7\sqrt{2}$, or 9.898. **3.** $5\sqrt{2}$, or 7.070. **5.** $2\frac{2}{3}\sqrt{3}$, or 12.701.

7. $1\frac{6}{15}\sqrt{5}$, or 2.385. **9.** $1\frac{3}{6}\sqrt{6}+2\sqrt{2}$, or 8.134.

11. $\frac{7}{2}\sqrt{2}-\frac{1}{3}\sqrt{3}$. **15.** $\sqrt{2x}+x\sqrt{b}$. **19.** $2\sqrt[5]{3}$.

13. $3\sqrt[3]{3}$. **17.** $(3-2a-b)\sqrt{x}$. **21.** $(3x+xy)\sqrt[4]{x}$.

23. $\frac{15x}{2}\sqrt[3]{2x}$. **25.** $\frac{10-4x}{5x}\sqrt{5x}+\frac{9}{4x}\sqrt[3]{4x}$.

27. $(x+y-1)\sqrt{xy}$.

Exercise 38. Page 90

1. $2\sqrt{5}$, or 4.472. **5.** $6\sqrt[3]{2}$, or 7.560. **9.** $\sqrt{15a}$.

3. $5\sqrt[3]{2}$, or 6.300. **7.** $6\sqrt{3}$, or 10.392. **11.** $2a\sqrt{5b}$.

13. $15\sqrt{2}$. **19.** $6\sqrt[3]{15}$. **25.** $2xy\sqrt[4]{3xy}$. **31.** $15x\sqrt{x}$.

15. $2x\sqrt{3}$. **21.** $250x$. **27.** $x\sqrt{15}$. **35.** $1+\sqrt{15}$.

17. $24a$. **23.** x^2+y. **29.** $.3\sqrt{2}$. **37.** $21-8\sqrt{5}$.

39. $12\sqrt{5}-2\sqrt{35}+3\sqrt{7}-18$.

41. $3+4y\sqrt{3}+4y^2$. **45.** $36-2a$. **49.** $y-x$.

43. 17. **47.** $12-ax$. **51.** $6\sqrt[3]{x^2}+2\sqrt[3]{x}-4$.

53. $2x+7-2\sqrt{x^2+7x+10}$.

55. Zero. **59.** $\sqrt[3]{864hw}$. **63.** $\sqrt[4]{405ab}$.

57. $\sqrt{54h}$. **61.** $\sqrt[3]{108c^3dh}$.

Exercise 39. Page 92

1. $\sqrt{3}$, or 1.732. **9.** $\frac{2}{3}\sqrt[3]{9}$, or 1.387. **15.** $\frac{\sqrt{10}}{2}$.

3. $\frac{3}{2}\sqrt{2}$, or 2.121. **11.** $\frac{3}{b^2}\sqrt{2ab}$. **17.** $7\sqrt{2y}$.

5. $\frac{3}{2}\sqrt[3]{20}$, or 4.071. **13.** $\frac{a}{2}\sqrt[3]{4}$. **19.** $\frac{x\sqrt{2xy}}{y}$.

7. $\frac{5}{6}\sqrt{15}$, or 3.228.

21. $\frac{\sqrt[6]{5^5\,xy^4}}{5\,y}$.

23. $\frac{5}{2}\sqrt[3]{4} - \frac{3}{2}\sqrt[3]{2}$.

25. $\frac{2}{5}\sqrt{5}$, or .894.

27. $3\sqrt{2}$, or 4.242.

29. $\frac{3}{2^3 5}\sqrt{35}$, or .634.

31. $\sqrt[3]{12}$, or 2.289.

33. $\frac{\sqrt{30}}{30}$, or .183.

35. $60\sqrt{7}$, or 158.76.

37. $-5\sqrt{2} - 5\sqrt{3}$, or -15.73.

39. $\frac{6 + 2\sqrt{2}}{7}$, or 1.261.

41. $\frac{8 - 3\sqrt{6}}{10}$, or .0653.

43. $\frac{a}{18\,b}\sqrt{3\,b}$.

45. $\frac{(2 - x)(\sqrt{x - 1} + 2)}{x - 5}$.

47. $\sqrt[3]{10}$.

49. $\frac{\sqrt{3\,a + 3\,b + 2\sqrt{a + b}}}{a + b}$.

51. $\frac{4}{1}\frac{1}{3}\frac{2}{3}$.

53. $\frac{\sqrt{6\,a^2 + 6\,ab} + \sqrt{6\,ab}}{a}$.

55. $\frac{\sqrt[3]{4}}{2} + \frac{\sqrt[3]{2}}{3}$.

57. $-\frac{\sqrt[3]{4\,ab^2}}{2\,ab}$.

59. $\frac{2\sqrt[3]{6\,x^2}}{3\,x} - \frac{5\sqrt[3]{2\,y^2 z}}{y}$.

Exercise 40. Page 94

1. $\sqrt[6]{72}$.

3. $\sqrt[6]{27 \cdot 49}$.

5. $\frac{\sqrt[6]{27\,x^3 y^4}}{y}$.

7. $\frac{\sqrt{1372}}{14}$.

9. $2\sqrt[6]{2^4 3^5}$.

11. $2\,y\sqrt[6]{2\,y}$.

13. $\frac{1}{3}\sqrt[6]{x^4 y}$.

15. $\frac{1}{2}\sqrt{32}$.

17. $\frac{\sqrt[6]{72\,x^5}}{x^2}$.

19. $\frac{2\sqrt[12]{5^9(4\,x)}}{5}$.

21. $\frac{\sqrt[4]{8\,ax^3}}{2\,a}$.

23. $\frac{a\sqrt[4]{3} + 2\sqrt[4]{3\,a^2 b^2}}{a - 4\,b}$.

Exercise 41. Page 95

1. 8.

3. $3\sqrt[3]{3}$.

5. $\sqrt[4]{6}$.

7. $\sqrt[3]{2}$.

9. $16\,b\sqrt[3]{b}$.

11. $\sqrt[4]{ab^2 c}$.

13. $a\sqrt{2}$.

15. $2\,a\sqrt[4]{5}$.

17. $\sqrt[12]{3}$.

19. $9\,a^4 m\sqrt[3]{m}$.

21. $4\,ab\sqrt[3]{a^2 c^2}$.

Exercise 42. Page 95

1. 81.

3. $\frac{1}{2^5}$.

5. .00001.

7. 3.

9. 2.

11. 137.

13. $\frac{1}{3}\sqrt{3}$, or .577.

15. 12.

17. $\frac{1}{4}$.

19. $50\sqrt{2}$, or 70.7.

21. $\frac{\sqrt{5}}{5}$, or .447.

23. $\frac{1}{4}$.

25. $\frac{3}{5}$.

27. $\frac{2}{3}\sqrt{15}$, or 2.582. **31.** $\frac{8-3\sqrt{6}}{10}$, or .0653. **35.** $\sqrt{5}$, or 2.236.

29. $\sqrt{3}+\sqrt{2}$, or 3.146. **33.** $\frac{6}{5}\sqrt{15}$, or 4.648. **37.** $16\,a^4b^4$.

39. $\frac{x^2}{5\,a}$. **47.** $16\,a^{\frac{1}{2}}$. **55.** $\frac{c^{mn}}{d^nx^n}$. **61.** $2\,xy^2$.

63. $-2\,yz\sqrt[3]{z^2}$.

41. x^3y^6. **49.** $\frac{1}{x^4y^2}$.

43. x^9y^4. **51.** $3\,y^{\frac{2}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$. **57.** $\frac{27\,x^3}{64\,y}$. **65.** $\frac{1}{2}\,w$.

67. $2\,y\sqrt[5]{xy}$.

45. $a^{\frac{2}{3}}$. **53.** $\frac{1}{5\,w^2y^2}$. **59.** xy^2-x^2y. **69.** $2\,c^2x\sqrt{2\,x}$.

71. $\frac{\sqrt{5}}{5}$. **73.** $\frac{4\,w^3}{5\,x}\sqrt{5}$. **75.** $\frac{x^k}{3\,y}\sqrt[4]{375\,x^{2k}y}$.

77. $\frac{21\sqrt{2}+2\sqrt{3}}{6}$. **87.** $xy^2\sqrt[3]{9\,x^2}$.

89. $x\sqrt[3]{y}$.

79. $(2\,z+3\,x)\sqrt[4]{3\,xy^2}-y\sqrt{5\,x}$. **91.** $\frac{\sqrt{2+x-x^2}-\sqrt{2-x}}{x}$.

81. $\frac{\sqrt{3\,b-25\,b^2}}{b}$.

93. $\frac{\sqrt{3\,a}-\sqrt{3\,b}+2\sqrt{a}-2\sqrt{b}}{a-b}$.

83. $\frac{\sqrt{x^2+x-6}}{x-2}$.

85. $9\,x^2$. **95.** $\frac{15\sqrt{x}-5\,x}{9-x}$.

97. $\sqrt[15]{a}$. **101.** $x\sqrt{5\,x}$. **105.** $\sqrt{3}$. **109.** $\sqrt[6]{a^3b^2}$.

99. $\sqrt{10\,z}$. **103.** $\sqrt{2}$. **107.** $\sqrt{3}$. **111.** $2\,y\sqrt[12]{2\,y^5}$.

113. $yz\sqrt[6]{16\,y^3z^3}$. **121.** No solution.

115. $\frac{\sqrt[6]{(x-y)^2(x+y)^5}}{x+y}$. **123.** $x=4,\ y=5$.

125. $x=-\frac{417}{139},\ y=\frac{695}{139}$.

119. $x=-2,\ y=6$. **127.** $x=2;\ y=-3;\ z=2$.

129. $\frac{5}{7}$.

131. \$3000 at 3%; \$4500 at 4%.

133. 40 miles, and 20 miles per hour. **135.** 2, 3 and -1.

Exercise 43. Page 100

1. ± 5. **5.** $\pm\frac{\sqrt{35}}{5}$, or ± 1.1832.

3. $\pm\frac{\sqrt{15}}{5}$, or $+.7746$. **7.** $\pm\frac{6\sqrt{21}}{7}$, or $+3.928$.

9. $\pm \dfrac{\sqrt{3\,ab - 5\,ac}}{2\,a}$.

11. $\pm \dfrac{\sqrt{6\,bc - 15\,ab}}{3\,b}$.

13. $\pm \dfrac{\sqrt{(b + 5\,d)(2 - c)}}{2 - c}$.

15. $\pm \dfrac{\sqrt{2\,gS}}{g}$.

Exercise 44. Page 101

1. $x = -5$; $x = 2$.
3. $x = 1$; $x = -\frac{3}{2}$.
5. $x = 0$; $x = \frac{4}{3}$.
7. $w = \pm 3$.
9. $z = \pm \frac{2}{5}$.
11. $y = -\frac{2}{5}$.
13. $x = -2$; $x = \frac{5}{3}$.
15. $x = \pm 12$.
17. $x = 0$; $x = -\frac{3}{5}$.
19. $z = 5$; $z = -1$.
21. $x = 2\,a$; $x = 3\,a$.
23. $w = c$; $w = \frac{3}{2}\,c$.
25. $x = \pm 2$.
27. $x = \frac{3}{4}$; $x = -\frac{2}{5}$.

Exercise 45. Page 103

1. 3; -8.
3. 9; -7.
5. $5 \pm \sqrt{2}$: or, 6.414; 3.586.
7. $\dfrac{-9 \pm 3\sqrt{5}}{2}$: or, -1.146; -7.854.
9. $-\frac{5}{2}$; $\frac{5}{3}$.
11. $\frac{1}{3}$; $-\frac{5}{2}$.
13. $\dfrac{5 \pm \sqrt{15}}{2}$: or $\begin{cases} 4.4365. \\ .5635. \end{cases}$
15. $\dfrac{5 \pm \sqrt{33}}{4}$: or $\begin{cases} 2.686. \\ -.186. \end{cases}$
17. $\dfrac{-3 \pm \sqrt{15}}{3}$: or $\begin{cases} -2.291. \\ .291. \end{cases}$
19. $\dfrac{-7 \pm \sqrt{65}}{8}$: or $\begin{cases} -1.883. \\ .133. \end{cases}$
21. $\dfrac{4 \pm \sqrt{6}}{2}$: or $\begin{cases} 3.2245. \\ .7755. \end{cases}$
23. $\begin{cases} -5. \\ -\frac{1}{2}. \end{cases}$
25. $\dfrac{-2 \pm \sqrt{34}}{10}$: or $\begin{cases} -.783. \\ .383. \end{cases}$
27. $\frac{4}{5}$; $-\frac{3}{5}$.
29. $-\frac{4}{5}$; $\frac{1}{5}$.
33. $-b \pm \sqrt{b^2 - c}$.
35. $\dfrac{4 \pm \sqrt{16 + 2\,b}}{2}$.
37. $\dfrac{-2 \pm \sqrt{4 - ac}}{a}$.
39. $-m$; $(-3 + m)$.
41. a; $-b$.
43. a; $\dfrac{b}{a}$.

Exercise 46. Page 106

1. $\begin{cases} -\frac{2}{3}. \\ -\frac{3}{2}. \end{cases}$
3. $\begin{cases} \frac{3}{5}. \\ -2. \end{cases}$
5. $\begin{cases} -2. \\ -\frac{1}{3}. \end{cases}$
7. $\dfrac{-3 \pm \sqrt{5}}{2}$: or $\begin{cases} -2.618. \\ -.382. \end{cases}$
9. $\begin{cases} 1. \\ -\frac{2}{5}. \end{cases}$

11. $4 \pm 3\sqrt{2}$: or $\begin{cases} 8.242. \\ -.242. \end{cases}$

13. $\frac{1 \pm \sqrt{13}}{2}$: or $\begin{cases} 2.303. \\ -1.303. \end{cases}$

15. $\frac{1 \pm \sqrt{21}}{4}$: or $\begin{cases} 1.396. \\ -.896. \end{cases}$

17. $\pm \frac{\sqrt{10}}{2}$: or $\begin{cases} 1.581. \\ -1.581. \end{cases}$

19. $\begin{cases} .4. \\ -.25. \end{cases}$

21. $\begin{cases} \frac{3}{2}. \\ -\frac{1}{3}. \end{cases}$

23. $\frac{-3 \pm \sqrt{89}}{40}$: or $\begin{cases} -.311. \\ .161. \end{cases}$

25. $\begin{cases} \frac{5}{3}. \\ -1. \end{cases}$

27. $\frac{-3 \pm \sqrt{33}}{4}$: or $\begin{cases} -2.186. \\ .686. \end{cases}$

29. $\begin{cases} .5. \\ -.8. \end{cases}$

31. $\frac{d}{2}(1 \pm \sqrt{11})$: or $\begin{cases} 2.1585\,d. \\ -1.1585\,d. \end{cases}$

33. $-a \pm \sqrt{a^2 + 3d}$.

35. $(-b \pm \sqrt{b^2 - 3c}) \div c$.

37. $\begin{cases} -1. \\ -a. \end{cases}$

39. $\frac{b \pm \sqrt{b^2 + 4ac}}{2a}$.

Exercise 47. Page 108

1. $3i$. **5.** $7ix$. **9.** $5ai\sqrt{5}$. **13.** $\frac{1}{2}i$.

3. $12i$. **7.** $2i\sqrt{2}$. **11.** $8ix^3\sqrt{2}$. **15.** $5i\sqrt{6}$.

17. $\frac{2}{5}i$. **21.** $\frac{1}{2}(3 + 3i\sqrt{2})$. **25.** $7i - 9$.

19. $\frac{3}{7}i$. **23.** $-2 + i\sqrt{6}$. **27.** $9 - 15i$.

29. $-\sqrt{15} - 3$. **31.** $\pm 5i$; $\pm 8i$; $\pm i\sqrt{15}$; $\pm \frac{i}{4}\sqrt{2}$.

Exercise 48. Page 109

3. $w = 1 \pm 2i$.

5. $z = \frac{3 \pm i\sqrt{111}}{12}$.

7. $x = 1 \pm 3i$.

9. $x = \frac{1 \pm i\sqrt{14}}{5}$.

11. $x = \frac{3 \pm i\sqrt{6}}{3}$.

Exercise 49. Page 109

1. $\begin{cases} \frac{1}{2}. \\ -3. \end{cases}$

3. $\begin{cases} 2. \\ 1. \end{cases}$

5. $\frac{5 \pm \sqrt{97}}{6}$: or $\begin{cases} 2.475. \\ -.808. \end{cases}$

7. $\frac{1 \pm i\sqrt{14}}{5}$.

9. $\frac{5 \pm \sqrt{89}}{16}$: or $\begin{cases} .902. \\ -.277. \end{cases}$

11. $\frac{-9 \pm \sqrt{21}}{10}$: or $\begin{cases} -1.358. \\ -.442. \end{cases}$

13. $\frac{-3 \pm i\sqrt{55}}{4}$.

15. $\begin{cases} 2. \\ -4. \end{cases}$

17. $-3 \pm \sqrt{2}$: or $\begin{cases} -1.586. \\ -4.414. \end{cases}$

19. $\begin{cases} -2. \\ -\frac{1}{6}. \end{cases}$

21. $\begin{cases} 5. \\ -8. \end{cases}$

23. $\frac{-a \pm \sqrt{a^2 + 2gS}}{g}$.

25. $\begin{cases} -\frac{1}{2}. \\ \frac{1-b}{2}. \end{cases}$

27. $\begin{cases} 4. \\ 1. \end{cases}$

29. 16 ft. by 23 ft.

31. 7 ft. and 2 ft.

33. Either $\frac{2}{3}$ or $\frac{3}{2}$.

35. 5 ft. and 8 ft.

37. 7.122 yds.

Exercise 51. Page 117

1. $x = 4\frac{1}{2}$.

3. $x = \frac{7}{3}$.

5. $x = 2$; $x = 1$.

7. No real roots.

9. $x = 2$; $x = 2\frac{1}{2}$.

11. $x = 0$; $x = \frac{3}{8}$.

13. $x = 1\frac{1}{2}$.

15. $z = 2.4$; $z = -.8$.

17. $y = 2.3$; $y = -.3$.

19. $x = 2.6$; $x = -2$.

Exercise 53. Page 120

7. $f(-3) = -11$.

9. $f(2a) = 6a - 2$.

11. $G(1) = 1$; $G(-3) = -27$; $G(\frac{1}{4}) = \frac{5}{8}$.

13. $f(1) = 2$; $5f(-3) = \frac{10}{3}$; $[f(b)]^2 = \frac{b^2 + 2b + 1}{b^2}$; $f(a-1) = \frac{a}{a-1}$.

Exercise 54. Page 122

1. Discr. $= 16$; roots are real, unequal, rational.

3. Discr. $= 49$; roots are real, unequal, rational.

5. Discr. $= 133$; roots are real, unequal, irrational.

7. Discr. $= -48$; roots are imaginary.

9. Discr. $= -11$; roots are imaginary.

11. Discr. $= -1.35$; roots are imaginary.

13. Discr. $= 109$; roots are real and unequal.

19. $k = \pm 2$.

21. $k = 0$.

23. $k = -1.19$; $k = .75$.

25. $k = \pm \frac{3}{2}i$.

27. $k = 0$; $k = -\frac{8}{9}$.

Exercise 55. Page 124

1. Sum $= 5$; prod. $= 6$.
3. Sum $= 4$; prod. $= 6$.
5. Sum $= 0$; prod. $= -\frac{1}{4}$.
7. Sum $= \frac{10}{3}$; prod. $= \frac{25}{9}$.
9. Sum $= \frac{5}{2}$; prod. $= 1$.
11. $x = \frac{3}{2}$.
13. $h = \frac{3}{13}$.
15. $h = \frac{1}{4}$.
17. $h = -1$.
19. $h = \pm 2$.
21. $h = -1$.
23. $h = \pm 10$.

Exercise 56. Page 125

1. $x^2 - 8x + 15 = 0$.
3. $12x^2 + x - 6 = 0$.
5. $x^2 - (a + b)x + ab = 0$.
7. $x^2 - 2x - 1 = 0$.
9. $x^2 - 4x + 29 = 0$.
11. $x^2 - 6x + 14 = 0$.
13. $4x^2 - 8x + 7 = 0$.

Exercise 57. Page 127

1. $x = \pm 1$; $x = \pm 1.414$.
3. $x = \pm 2$; $x = \pm .707$.
5. $x = \pm 2$; $x = \pm 2i$.
7. $y = \pm 1$; $y = \pm \frac{1}{2}$.
9. $x = \pm 1$; $x = -5$; $y = -7$.
11. $x = \pm 2$; $x = 3$; $x = -1$.
13. $w = \pm 2.236$; $w = \pm \frac{i}{2}\sqrt{6}$.
15. $x = 1$; $x = -2$; $x = 5.372$; $x = -.372$.
17. $x = 1$; $x = -1$; $x = -2$; $x = -4$.
19. $y = 8$; $y = 4$; $y = \frac{-1 \pm i\sqrt{41}}{6}$.

Exercise 58. Page 129

1. 7.
3. No solution.
5. 16.
7. .707.
9. 1; 4.
11. -1.
13. 0; 1.732.
15. 3; 0.
17. 5.
19. No solution.
21. No solution.
23. -5.
25. $\frac{2}{3}$.

Supplementary Exercise 59. Page 130

1. 4.
3. No solution.
5. ± 243.
7. -32.
9. ± 2.
11. 81.
13. 5; -3.
15. 625.
17. 16.
19. $3a$; 0.
21. $3a$.

23. $s = \frac{v^2}{2g}$; $g = \frac{v^2}{2s}$.

25. $h = \pm \frac{\sqrt{l^2 - \pi^2 r^4}}{\pi r}$.

31. $h = 1$; $k = -1$.

33. (*a*) $h = \pm \frac{1}{2}$. (*b*) $h = -2$; $h = \frac{1}{3}$.

47. 3; $\frac{-3 \pm 3i\sqrt{3}}{2}$.

49. $\pm 2\sqrt{2}$; $\pm 2i\sqrt{2}$.

51. $\pm \frac{2}{3}$; $\pm \frac{2i}{3}$.

53. $-\frac{1}{5}$; $\frac{1 \pm i\sqrt{3}}{10}$.

55. $\pm \frac{1}{2}$; $\pm \frac{i}{2}$.

57. *Fourth roots of* 81 : $3, -3, 3i, -3i$.
Fourth roots of 625 : $5, -5, 5i, -5i$.

Exercise 60. Page 135

1. $(x = 0, y = -3)$; $(x = 2.4, y = 1.8)$.

3. $(x = 1.35, y = 1.8)$; $(x = -1.35, y = 1.8)$.

5. $(x = -5, y = -3)$.

7. $(x = 4.5, y = 2.2)$; $(x = -4.5, y = -2.2)$; $(x = -4.2, y = 2.8)$; $(x = 4.2, y = -2.8)$.

9. $(x = 2.1, y = 1.4)$; $(x = 2.1, y = -1.4)$; $(x = -2.1, y = 1.4)$; $(x = -2.1, y = -1.4)$.

Exercise 61. Page 136

1. $(x = 3, y = 4)$; $(x = 0, y = 5)$.

3. $(x = 6 + 4i\sqrt{3}, y = 8 - 3i\sqrt{3})$;
$(x = 6 - 4i\sqrt{3}, y = 8 + 3i\sqrt{3})$.

5. $(c = 2, d = -5)$; $(c = -5, d = 2)$.

7. $(x = \frac{3}{2}, y = \frac{1}{2})$; $(x = -\frac{1}{2}, y = -\frac{3}{2})$.

9. $(x = 5, y = 0)$; $(x = -\frac{25}{3}, y = -\frac{8}{3})$.

11. $\left(x = \frac{7 + i\sqrt{77}}{6}, y = \frac{-8 + i\sqrt{77}}{6}\right)$;
$\left(x = \frac{7 - i\sqrt{77}}{6}, y = \frac{-8 - i\sqrt{77}}{6}\right)$.

13. $(x = 3, y = 0)$; $(x = -\frac{9}{8}, y = \frac{33}{8})$.

Exercise 62. Page 138

1. $(x = 6.634;\ y = 2.236)$; $(x = 6.634, y = -2.236)$;
$(x = -6.634;\ y = 2.236)$; $(x = -6.634, y = -2.236)$.

3. $(x = 2i\sqrt{2}, y = 3\sqrt{2})$; $(x = 2i\sqrt{2}, y = -3\sqrt{2})$;
$(x = -2i\sqrt{2}, y = 3\sqrt{2})$; $(x = -2i\sqrt{2}, y = -3\sqrt{2})$.

5. $(u = \frac{3}{2}, v = 2)$; $(u = -\frac{3}{2}, v = 2)$;
$(u = -\frac{3}{2}, v = -2)$; $(u = \frac{3}{2}, v = -2)$.

7. $(x = .935, y = .866)$; $(x = .935, y = -.866)$;
$(x = -.935, y = .866)$; $(x = -.935, y = -.866)$.

Exercise 63. Page 139

1. $(x = 14, y = -2)$; $(x = -14, y = 2)$;
$(x = 4, y = \frac{1}{2})$; $(x = -4, y = -\frac{1}{2})$.

3. $(x = \frac{1}{2}, y = -3)$; $(x = -\frac{1}{2}, y = 3)$;
$(x = 2, y = -2)$; $(x = -2, y = 2)$.

5. $(x = 4, y = 3)$; $(x = -4, y = -3)$;
$(x = .707, y = -4.949)$; $(x = -.707, y = 4.949)$.

7. $(x = \frac{1}{2}, y = -\frac{1}{2})$; $(x = -\frac{1}{2}, y = \frac{1}{2})$;
$(x = 1.414, y = .236)$; $(x = -1.414, y = -.236)$.

Exercise 64. Page 140

1. $(x = \frac{1}{2}, y = -1)$; $(x = -\frac{1}{2}, y = 1)$;
$(x = 1.414, y = .471)$; $(x = -1.414, y = -.471)$.

3. $(x = 4, y = 1)$; $(x = -4, y = -1)$;
$(x = 14, y = -4)$; $(x = -14, y = 4)$.

5. $(x = 1, y = 7)$; $(x = -1, y = -7)$;
$(x = 7, y = 1)$; $(x = -7, y = -1)$.

7. $(x = 1, y = 0)$; $(x = -1, y = 0)$;
$\left(x = i\sqrt{2}, y = \frac{3i}{2}\sqrt{2}\right)$; $\left(x = -i\sqrt{2}, y = -\frac{3i}{2}\sqrt{2}\right)$.

9. $(x = 4, y = 5)$; $(x = -4, y = -5)$;
$(x = 5.196, y = 1.732)$; $(x = -5.196, y = -1.732)$.

11. $(x = 2, y = 1)$; $(x = -4, y = -2)$;
$(x = 3, y = 2)$; $(x = -\frac{3}{2}, y = -1)$.

13. $(x = 0, y = 3)$; $(x = 3, y = 0)$.

15. $(x = 4, y = 2)$; $(x = -4, y = -2)$;
$(x = -4, y = 2)$; $(x = 4, y = -2)$.

17. $(x = 0, y = -4)$; $(x = -4, y = 0)$;
$(x = 2.158, y = -1.158)$; $(x = -1.158, y = 2.158)$.

19. $(x = 3, y = 1)$; $(x = 1, y = 3)$;
$(x = -1, y = -3)$; $(x = -3, y = -1)$.

21. $(x = -1, y = -\frac{2}{3})$; $(x = -\frac{2}{3}, y = -1)$;
$(x = -.138, y = .804)$; $(x = 804, y = -.138)$.

23. $(x = 3, z = 1)$; $(x = 5, z = -1)$;
$(x = -5, z = 4)$; $(x = 3, z = -4)$.

Exercise 65. Page 142

1. $(x = \frac{3}{2}, y = 4)$; $(x = 2, y = 3)$.

3. $(x = -\frac{1}{2}, y = 1)$; $(x = \frac{1}{2}, y = -1)$;
$(x = 1.414, y = .471)$; $(x = -1.414, y = -.471)$.

5. $(x = 0, y = 0)$; $(x = 1, y = 1)$;
$(x = -1, y = 1)$; $(x = -\frac{1}{5}, y = \frac{3}{5})$.

7. $(x = \frac{3}{2}, y = \frac{2}{3})$; $(x = -\frac{3}{2}, y = \frac{2}{3})$;
$(x = \frac{3}{2}, y = -\frac{2}{3})$; $(x = -\frac{3}{2}, y = -\frac{2}{3})$.

9. $(x = 4, y = 3)$; $(x = 4, y = -3)$; $(x = -4, y = 3)$;
$(x = -4, y = -3)$.

11. $(x = \frac{1}{7}, y = -\frac{1}{5})$; $(x = \frac{1}{5}, y = -\frac{1}{7})$.

13. -10 and 1.

15. Rectangle: 8 ft. by 13 ft.; side of square: 6 ft.

17. The numbers are (2 and 3) or $(-8$ and $-2)$.

19. The numbers are either $(\frac{5}{2}$ and 24) or $(-\frac{5}{2}$ and $-24)$.

21. The original member is 36.

23. { Rate of current is 2 miles per hour.
Rate at which boys can row is 4 miles per hour.

25. The diameters are 35 inches and 30 inches.

27. $k = \pm .707$. 29. $k = -\frac{3}{4}$. 31. $c^2 = a^2m^2 - b^2$.

33. $(x = 4, y = 3.464)$; $(x = 4, y = -3.464)$;
$(x = -\frac{5}{2}, y = \frac{3}{2})$; $(x = -\frac{5}{2}, y = -\frac{3}{2})$.

35. $(u = 2, y = 4.472)$; $(u = 2, y = -4.472)$;
$(u = -\frac{5}{3}, y = 3)$; $(u = -\frac{5}{3}, y = -3)$.

37. $(a = \frac{7}{3}, b = \frac{1}{3})$.

39. $(x = -3, y = -2, z = 1)$; $(x = \frac{7}{4}, y = \frac{11}{4}, z = \frac{27}{8})$.

41. $(x = 1.732, y = 2, z = i\sqrt{5})$; $(x = -1.732, y = 2, z = i\sqrt{5})$;
$(x = 1.732, y = -2, z = i\sqrt{5})$; $(x = -1.732, y = -2, z = i\sqrt{5})$;
$(x = 1.732, y = -2, z = -i\sqrt{5})$;
$(x = -1.732, y = -2, z = -i\sqrt{5})$;
$(x = 1.732, y = 2, z = -i\sqrt{5})$; $(x = -1.732, y = 2, z = -i\sqrt{5})$.

43. $(x = a - 1;\ y = a + 1)$; $(x = a + 1;\ y = a - 1)$.

45. $(x = 2a - b,\ y = a - 2b)$; $\left(x = -\dfrac{a + 7b}{8},\ y = \dfrac{25a - 17b}{8}\right)$.

47. $(x = 125,\ y = -1)$; $(x = -125,\ y = 1)$;
$(x = 27,\ y = -343)$; $(x = -27,\ y = 343)$.

Review Exercise 66. Page 145

1. $x = 1,\ x = -\frac{4}{3}$. **3.** $x = \pm\frac{3}{2}$.

11. $h = 0;\ h = 4$. **13.** $h = \frac{3}{4}$. **15.** $h = \pm\frac{1}{3}$.

17. $f(-3) = -\dfrac{22}{3}$. $f(2) = \dfrac{7}{2}$. $f\left(\dfrac{a}{b}\right) = \dfrac{3a^2 - 5b^2}{ab}$.

19. $(x = 3.15,\ y = .6)$; $(x = 3.15,\ y = -.6)$;
$(x = -3.15,\ y = .6)$; $(x = -3.15,\ y = -.6)$.

21. $x = -1$. **25.** $x = -2;\ x = -\frac{1}{6}$.

23. $x = \pm .866;\ x = \pm\dfrac{i}{3}$.

27. $(x = \frac{1}{4},\ y = \frac{1}{3})$; $(x = -\frac{1}{4},\ y = \frac{1}{3})$;
$(x = \frac{1}{4},\ y = -\frac{1}{3})$; $(x = -\frac{1}{4},\ y = -\frac{1}{3})$.

29. Either 12, or $\frac{1}{4}$. **31.** $x = \frac{2}{3}$.

33. (*a*) $x = 7.583;\ x = -1.583$. (*b*) $x = 5.828;\ x = -.172$.

Exercise 67. Page 148

1. $\dfrac{3x}{2}$. **5.** $\frac{7}{200}$. **9.** $x = \frac{5}{6}$.

7. $x = 20$. **11.** $x = 3.775$.

3. $\dfrac{1}{12a + 2}$.

13. 6 and 9 inches. **15.** The parts are $26\frac{3}{5}$ and $48\frac{2}{5}$.

17. The other sides are $\frac{140}{11}$ and $\frac{80}{11}$ inches.

19. $527\frac{7}{9}$ square inches.

21. 24 feet. **23.** -35. **25.** $\dfrac{ab^2}{2}$. **27.** $\frac{1250}{9}$.

29. $\dfrac{a}{2}$. **31.** ± 25. **33.** ± 6.

Supplementary Exercise 68. Page 151

1. $\dfrac{2}{5} = \dfrac{m}{x};\ \dfrac{m}{2} = \dfrac{x}{5};\ \dfrac{2 + m}{m} = \dfrac{5 + x}{5};\ \dfrac{2 - m}{m} = \dfrac{5 - x}{x};$
$\dfrac{2 + m}{2 - m} = \dfrac{5 + x}{5 - x}$.

3. $\frac{3}{c} = \frac{7}{y}$; $\frac{7}{3} = \frac{y}{c}$; $\frac{10}{7} = \frac{c+y}{y}$; $-\frac{4}{7} = \frac{c-y}{y}$; $-\frac{5}{2} = \frac{c+y}{c-y}$.

5. $\frac{x+c}{c} = \frac{y+d}{d}$.

Exercise 69. Page 154

1. $w = \frac{ky}{z}$.

3. $y = 3x$.

5. $y = -\frac{18}{x^2}$.

7. $5y - 12x = 3$.

9. $w = \frac{k}{d^2}$.

15. $y = \frac{7}{3}$.

17. 576 feet.

19. The new illumination is $\frac{1}{4}$ of the old.

21. $y = 8x$.

23. (*a*) $\sqrt[3]{\frac{4489}{8649}}$; (*b*) $\sqrt[3]{\frac{20022.25}{8649}}$.

25. $x = 20$, $y = -\frac{40}{3}$, $z = \frac{80}{3}$.

27. ($x = .1826$, $y = .9130$, $z = .3652$), or ($x = -.1826$, $y = -.9130$, $z = -.3652$).

29. 2000 acres, 500 acres, and 250 acres.

Exercise 70. Page 159

1. 9th is 34; 61st is 242.

3. 36th is -178.

5. $l = 69$; $S = 511$.

7. $l = -79$; $S = -657$.

9. $l = 2.76$; $S = 42.24$.

11. $d = 16$; $S = 5278$.

13. $d = \frac{33}{7}$; $S = 600$.

15. $d = \frac{5}{9}$; $l = 9$.

17. The means are $-\frac{2}{5}$, $-\frac{14}{5}$, $-\frac{26}{5}$, and $-\frac{38}{5}$.

19. The means are $\frac{33}{16}$, $\frac{21}{8}$, $\frac{51}{16}$, $\frac{15}{4}$, $\frac{69}{16}$, $\frac{39}{8}$, and $\frac{87}{16}$.

21. The means are

$$\frac{10h+k}{11}, \frac{9h+2k}{11}, \frac{8h+3k}{11}, \frac{7h+4k}{11}, \frac{6h+5k}{11}, \frac{5h+6k}{11},$$
$$\frac{4h+7k}{11}, \frac{3h+8k}{11}, \frac{2h+9k}{11}, \frac{h+10k}{11}.$$

23. 25th term.

25. 7.

27. 10,816.

29. 4959.

31. (*a*) \$2300; (*b*) \$41,400.

33. \$775.

Exercise 71. Page 163

1. 8th = 128. 12th = 2048.

3. 7th = -96.

5. $l = \frac{5}{32}$. $s = 39\frac{27}{32}$.

7. $l = .000005$. $s = 5.55555$.

9. $l = 4\,b^{28}$. $s = \dfrac{4 - 4\,b^{30}}{1 - b^2}$.

11. $l = -\frac{27}{4}$. $s = -\frac{91}{18}$.

13. $\dfrac{(1.04)^{\frac{1}{2}} - (1.04)^{27}}{1 - (1.04)^{\frac{1}{2}}}$.

15. $\dfrac{5(1.03)^{-4} - 5(1.03)^{-42}}{1 - (1.03)^{-2}}$.

17. $a = 50$; $n = 4$.

19. $a = 128$; $n = 8$.

21. 1st term is 2.

23. ratio $= \pm \frac{1}{3}$.

25. The means are (2, 4, 8, 16, and 32), or (-2, 4, -8, 16, and -32).

27. (*a*) The means are $\frac{1}{4}$ and 1. (*b*) The means are ($\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, 1, and 2), or ($-\frac{1}{8}$, $\frac{1}{4}$, $-\frac{1}{2}$, 1, and -2).

29. (*a*) 27. (*b*) $\begin{cases} \sqrt{xy}, \text{ if } x \text{ and } y \text{ are positive.} \\ -\sqrt{xy}, \text{ if } x \text{ and } y \text{ are negative.} \end{cases}$

31. Either ($\frac{4}{3}$, $\frac{4}{9}$, $\frac{4}{27}$) or ($-\frac{4}{3}$, $\frac{4}{9}$, $-\frac{4}{27}$).

Miscellaneous Problems on Progressions. Page 165

1. \$409.50. **3.** 244.8 ft. **5.** 10.74 gal.

7. (*a*) At the end of 10 years. (*b*) At the end of 20 years.

9. 36,785 units. **11.** 1st term is 1; common difference is 2.

13. $-\frac{1}{2}$. **15.** rate = 11.8%. **17.** 3, 4, 5, 6 and 7.

Exercise 72. Page 168

1. $\frac{1}{4}$, $\frac{1}{6}$, and $\frac{1}{8}$. **3.** $\frac{1}{2}$, $\frac{5}{8}$, $\frac{5}{6}$, and $\frac{5}{4}$. **5.** (*a*) $\frac{175}{16}$; (*b*) $\dfrac{2\,xy}{x + y}$.

Exercise 73. Page 170

1. 9. **5.** $\frac{50}{9}$. **9.** $\frac{2}{3}$. **13.** $\frac{73}{300}$. **17.** $\frac{1}{6}$.

3. $\frac{5}{6}$. **7.** 500 feet. **11.** $\frac{199}{999}$. **15.** $\frac{7}{11}$. **19.** $\frac{26651}{45000}$.

Exercise 75. Page 179

1. $x^5 + 5\,x^4y + 10\,x^3y^2 + 10\,x^2y^3 + 5\,xy^4 + y^5$.

3. $a^4 + 4\,a^3b^2 + 6\,a^2b^4 + 4\,ab^6 + b^8$.

5. $a^4 - 12\,a^3b + 54\,a^2b^2 - 108\,ab^3 + 81\,b^4$.

7. $r^3 - 6\,r^{\frac{5}{2}} + 15\,r^2 - 20\,r^{\frac{3}{2}} + 15\,r - 6\,r^{\frac{1}{2}} + 1$.

9. $a^8 + 4\,a^6x^2y + 6\,a^4x^4y^2 + 4\,a^2x^6y^3 + x^8y^4$.

11. $x^{\frac{3}{4}} + 3\,x^{\frac{1}{2}} + 3\,x^{\frac{1}{4}} + 1$.

13. $a^{15} - 45\,a^{14} + 945\,a^{13}$.

15. $\dfrac{1}{a^6} + \dfrac{6}{a^5y} + \dfrac{15}{a^4y^2}$.

17. $a^{16} - 8\,a^{15} + 30\,a^{14}$.

19. $\dfrac{a^8}{b^8} + 8\,\dfrac{a^6}{b^6} + 28\,\dfrac{a^4}{b^4}$.

21. $x^{36} + 36\,x^{33}y + 594\,x^{30}y^2$.

23. $x^{\frac{7}{2}} - 7\,x^3y^{\frac{1}{2}} + 21\,x^{\frac{5}{2}}y$.

25. 24. **27.** 5040. **29.** 60.

Exercise 76. Page 181

1. $56\,a^5x^3$. **3.** $165\,m^3n^8$. **5.** $1120\,x^8z^{12}$.

7. $\dfrac{n(n-1)(n-2)(n-3)(n-4)}{5!}x^{n-5}y^5$. **9.** $312\,x^{-11}y$.

11. $-14{,}784\,c^6$. **15.** $35\,x^3y^4$; $35\,x^4y^3$.

13. $700{,}000\,x^4$. **17.** 1.1046.

Exercise 77. Page 186

1. $x > \frac{7}{3}$. **3.** $x > -\frac{3}{2}$. **5.** $x > -4$. **7.** $x > \frac{5}{3}$. **9.** $x < \frac{13}{16}$.

Exercise 78. Page 187

1. $x > \frac{5}{7}$. **3.** $x > 2$, and $x < -4$. **5.** $x > \frac{3}{2}$, and $x < -2$.

7. Values of x between -2 and 3; that is, values of x for which $-2 < x < 3$.

9. Values of a for which $-\frac{4}{3} < a < \frac{4}{3}$. **11.** No values.

13. Roots are real and unequal when $k > -\frac{5}{12}$.
Roots are imaginary when $k < -\frac{5}{12}$.

15. Roots are real and unequal when $k > \frac{2}{3}\sqrt{5}$, or $k < -\frac{2}{3}\sqrt{5}$.
Roots are imaginary for values of k for which $-\frac{2}{3}\sqrt{5} < k < \frac{2}{3}\sqrt{5}$.

17. When the value of k satisfies $-\frac{4}{3} < k < \frac{4}{3}$.

19. (*a*) $-\dfrac{\sqrt{11}}{5} < m < \dfrac{\sqrt{11}}{5}$; (*b*) if $m > \dfrac{\sqrt{11}}{5}$ or $m < -\dfrac{\sqrt{11}}{5}$.

(*c*) $m = \pm\dfrac{\sqrt{11}}{5}$.

Exercise 79. Page 191

1. $i\sqrt{7}$. **7.** -1. **13.** $3 - 3i$. **19.** -75.

3. $5i$. **9.** -1. **15.** $-1 + 8i$. **21.** $-2\sqrt{15}$.

5. $7i\sqrt{2}$. **11.** i. **17.** $-18i$. **23.** $32i$.

25. $53 - 9i$. **31.** $-3i\sqrt{7} + 6$. **37.** $\frac{10}{63} + \frac{325}{63}i$.

27. 34. **33.** $-6i$. **39.** $\frac{5}{4}\sqrt{3} - \frac{5}{4}i$.

29. $5 - 6i$. **35.** $\frac{11}{10} - \frac{7}{10}i$. **41.** $-\frac{3}{2}i$.

43. $65 + 142i$. **45.** $\frac{5}{29} - \frac{2}{29}i$.

47. $\dfrac{3}{34} - \dfrac{5}{34}i$; $-\dfrac{i}{7}$; $\dfrac{6}{43} - \dfrac{i}{43}\sqrt{7}$.

49. $x = 3$, $y = 2$. **51.** $x = -\frac{3}{2}$, $y = -3$.

Exercise 82. Page 196

11. $3\sqrt{2}(\cos 45° + i \sin 45°)$. **15.** $5(\cos 0° + i \sin 0°)$.

13. $\sqrt{65}(\cos 119° 45' + i \sin 119° 45')$.

17. $\sqrt{41}(\cos 308° 40' + i \sin 308° 40')$.

19. $5(\cos 233° 7' + i \sin 233° 7')$. **21.** $(\cos 240° + i \sin 240°)$.

23. $1 + i = \sqrt{2}(\cos 45° + i \sin 45°)$;
$1 - i = \sqrt{2}\,[\cos(-45°) + i \sin(-45°)]$.

Exercise 83. Page 199

1. -81. **3.** $-128 - 128\,i$.

5. $100(\cos 73° 41' + i \sin 73° 41')$.

7. $2(\cos 12° + i \sin 12°)$; $2(\cos 84° + i \sin 84°)$;
$2(\cos 156° + i \sin 156°)$.
$2(\cos 228° + i \sin 228°)$; $2(\cos 300° + i \sin 300°)$.

9. $(\cos 0° + i \sin 0°)$ or 1; $(\cos 120° + i \sin 120°)$;
$(\cos 240° + i \sin 240°)$.

11. $4(\cos 45° + i \sin 45°)$; $4(\cos 225° + i \sin 225°)$.

13. $2\sqrt{2}(\cos 33° + i \sin 33°)$; $2\sqrt{2}(\cos 123° + i \sin 123°)$;
$2\sqrt{2}(\cos 213° + i \sin 213°)$; $2\sqrt{2}(\cos 303° + i \sin 303°)$.

15. $\sqrt[6]{12}(\cos 110° + i \sin 110°)$; $\sqrt[6]{12}(\cos 230° + i \sin 230°)$;
$\sqrt[6]{12}(\cos 350° + i \sin 350°)$.

17. $(\cos 45° + i \sin 45°)$; $(\cos 225° + i \sin 225°)$.

19. $3(\cos 45° + i \sin 45°)$; $3(\cos 135° + i \sin 135°)$;
$3(\cos 225° + i \sin 225°)$; $3(\cos 315° + i \sin 315°)$.

21. $x = 3$; $x = 3\,i$; $x = -3$; $x = -3\,i$.

23. $x = 3$; $x = .927 + 2.853\,i$; $x = -2.427 + 1.764\,i$;
$x = -2.427 - 1.764\,i$; $x = .927 - 2.853\,i$.

Exercise 84. Page 201

1. $\frac{5}{3}(\cos 285° + i \sin 285°)$. **3.** $\frac{3\sqrt{2}}{2}(\cos 185° + i \sin 185°)$.

Exercise 85. Page 203

1. $f(3) = 23$; $f(-2) = 3$; $f(a) = a^3 - 3a + 5$; $f(c^2) = c^6 - 3c^2 + 5$.

3. 92. **5.** Zero. **7.** $a^4 + 2a^2 + a - 4$.

9. Yes, because the polynomial has the value zero if $x = 1$.

11. Yes. **13.** Yes, for $(x^7 + y^7)$; no, for $(x^7 - y^7)$.

15. $m = 0$. **17.** $k = \frac{7}{64}$.

Exercise 86. Page 206

1. $\dfrac{x^3 - 3x + 5}{x - 2} = x^2 + 2x + 1 + \dfrac{7}{x - 2}$.

3. Quot. is $2x^2 + 8x + 18$. Remainder is 37.

5. Quot. is $x^3 - 3x^2 + 6x - 16$. Remainder is 47.

7. $f(2) = -1$; $f(-2) = -25$.

9. $f(3) = 330$; $f(-1) = 22$; $f(-2) = 120$. **11.** $x^2 + 9$.

13. $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

15. $x^4 - 2x^3 + 4x^2 - 8x + 16$.

17. $x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-2}x + c^{n-1}$.

Exercise 87. Page 208

7. $-.5$; 5.5. **9.** 2.4; -1.3; -2.3. **11.** -1.9; .35; 1.5.

13. .4. **15.** -2.9; -1.1; 1.1; 3.95.

Exercise 88. Page 212

In the problems 1 to 7 the roots are as follows:

1. 2; -5; -7. **5.** $\pm 3i$; $\pm \frac{4}{3}i$.

3. 0; 3; $\dfrac{5 \pm i\sqrt{7}}{2}$. **7.** $\pm 2i$; $-\frac{2}{3}$.

9. $x^4 - 3x^3 - 20x^2 + 84x - 80 = 0$.

11. $x^3 - 5x^2 + 5x + 3 = 0$.

13. $4x^4 - 29x^3 + 66x^2 - 261x + 270 = 0$.

15. $x^3 - 7x^2 + 20x - 24 = 0$. **17.** $x^3 - 6x^2 + 15x - 14 = 0$.

19. $x^4 - 8x^3 + 19x^2 + 16x - 42 = 0$.

21. $x^4 - 3x^3 - 12x^2 + 52x - 48 = 0$.

23. $-2 - 6i$. **25.** $8 + 3i$.

31. Other roots: ± 1.732.

Exercise 90. Page 216

1. $2X^3 + 3X^2 - 5X - 2 = 0$.

3. $2X^4 - 5X^2 - 2X - 1 = 0$.

5. $X^8 + X^7 - 3 = 0$.

7. Each root of $f(2X) = 0$ is one half of a corresponding root of $f(x) = 0$; etc.

Exercise 91. Page 218

1. Not more than two positive roots; not more than one negative root.

3. It has exactly one positive root, no negative root, and two imaginary roots.

5. At most one positive root; at most one negative root; at least two imaginary roots.

7. One negative and four imaginary roots.

9. Four imaginary roots.

11. At most one positive root; at most one negative root; at least two imaginary.

13. One negative and two positive.

Exercise 92. Page 220

1. *Upper* $= 3$. *Lower* $= -5$.

3. *Upper* $= 3$. *Lower* $= -3$.

5. *Upper* $= 7$. *Lower* $= 0$.

7. *Upper* $= 3$. *Lower* $= -3$.

Exercise 93. Page 222

1. $5;\ -1;\ -7$.

3. $-3;\ \frac{1}{2}(-5 \pm i\sqrt{3})$.

5. $\frac{2}{3};\ -.382;\ -2.618$.

7. 2 is a double root; $i\sqrt{2};\ -i\sqrt{2}$.

9. $3.646;\ -1.646;\ \frac{5}{2}$.

11. $-\frac{1}{8};\ \pm\frac{i\sqrt{2}}{2}$.

13. None. **15.** None. **17.** None. **19.** -3. **21.** None.

Exercise 94. Page 225

1. $y^3 + 3y^2 - 10y + 8 = 0$.

3. $y^4 - 10y^2 - 8y - 8 = 0$.

5. $y^3 + y^2 + 2y + 135 = 0$.

7. $\frac{45}{2};\ \frac{1}{2}(-1 \pm i\sqrt{7})$.

9. $-\frac{9}{4}$.

11. $-\frac{7}{3};\ \pm\frac{i}{6}$.

Exercise 95. Page 229

1. $2x^3 + 8x^2 + 3x - 75 = 0$.

3. $x^4 + 1.2x^3 - 2.46x^2 + 3.308x + 3.2381 = 0$.

5. $2x^3 - 3.12x^2 - 4.8776x + 2.098784 = 0$.

7. $4x^3 + 31x^2 + 75x + 50 = 0$.

9. $5x^4 - 43x^3 + 138x^2 - 196x + 105 = 0$.

Exercise 96. Page 233

In Problems 1 to 18, each root is given to the **nearest** *digit in the third decimal place.*

1. 4.383. **3.** -802. **5.** 1.172; 1.593.

2. -1.213. **4.** 1.414. **6.** 1.357; 1.692.

7. .802; $-.555$; -2.247; -1. **17.** 2.427; 2.757; -4.184.

9. 1.165; -2.374. **19.** 1.627.

11. $\frac{1}{2}$; .570. **21.** -3.756.

13. 2.817; 3.956; $-.630$; $-.142$. **23.** 1.21; .74; $-.46$.

15. 1.644; -1.644. **25.** 6.10.

27. $(x = 1.82;\ y = 1.31)$. $(x = 2.63;\ y = 4.92)$.

29. 2.68 inches, or 1.32 inches.

31. 1.09 inches. **33.** 5.15 inches.

Exercise 97. Page 235

1. 1; 5; -3. **5.** 2; -2; -6. **7.** -1; -2; -4.

Exercise 98. Page 239

1. -1; $2 + 4i\sqrt{3}$; $2 - 4i\sqrt{3}$. **5.** 1; -2; 1.414; -1.414.

3. 2; 2.414; $-.414$. **7.** 1; -2; 2.732; $-.732$.

Exercise 99. Page 240

1. $\log_3 9 = 2$; $\log_5 625 = 4$; $\log_{10} 100 = 2$; $\log_3 \frac{1}{27} = -3$.

5. $2 = \log_4 16$. $\frac{1}{3} = \log_8 2$. $4 = \log_5 625$. $3 = \log_{10} 1000$.
$3 = \log_5 125$. $2 = \log_2 4$. $2 = \log_5 25$. $-3 = \log_{10} .001$.
$\frac{1}{2} = \log_{100} 10$. $-1 = \log_{25} \frac{1}{25}$. $4 = \log_3 81$. $-2 = \log_{.01} .0001$

Exercise 101. Page 244

1. Char. = 3; mant. = .90309. **7.** Char. $= -2$; mant. = .78533.

3. Char. $= -2$; mant. = .60206. **9.** Char. $= -3$; mant. = .69897.

5. Char. = 2; mant. = .32940.

Exercise 103. Page 246

3. $\log 35{,}627 = 4.55178$. $\log .000035627 = -5 + .55178$.
$\log .035627 = -2 + .55178$. $\log 3.5627 = 0 + .55178$.

Exercise 105. Page 254

1. 26,200.	**7.** .0758.	**13.** 17.25.	**19.** 24,540.
3. 540.	**9.** 12,060.	**15.** 8464.	**21.** .01185.
5. 6.81.	**11.** .2162.	**17.** 3.787.	**23.** .001459.
25. .000000000009852.		**27.** 1436.	**29.** 269.1.

Exercise 106. Page 255

1. 2.29820.	**11.** 1.11850.	**21.** 1636.4.
3. 2.40878.	**13.** 2.17932.	**23.** 2.2499.
5. 9.13982 − 10.	**15.** 2.58458.	**25.** .044944.
7. 8.86666 − 10.	**17.** 33,230.	**27.** .0000020946.
9. 7.55302 − 10.	**19.** .07419.	**29.** 20.583.

31. 295.94. **33.** .00000001379.

Exercise 107. Page 257

Answers, when computed by ***four-place*** *logarithms:*

1. 2328.	**3.** .007335.	**5.** .8142.	**7.** 192.7.
9. .03204.	**11.** .00000002627.		**13.** 24.66.

Answers, when computed by ***five-place*** *logarithms:*

1. 2328.7.	**3.** .0073345.	**5.** .81422.	**7.** 192.79.
9. .032027.	**11.** .000000026266.		**13.** 24.650.

Exercise 108. Page 258

Answers, when computed by ***four-place*** *logarithms:*

1. 5,358,000.	**9.** .2826.	**17.** .5921.	**25.** .8578.
3. 12.77.	**11.** .4528.	**19.** .000001848.	**27.** 2.086.
5. .6030.	**13.** .00005176.	**21.** .9388.	**29.** 1657.
7. .0001913.	**15.** .8226.	**23.** 1.709.	**31.** − .007465.
33. 10.60.		**39.** − .003144.	
35. (*a*) 5354; (*b*) 3.472.		**41.** 37.64.	
37. (*a*) .7238; (*b*) .9484.		**43.** 1.247.	
		45. .09310.	

Answers, when computed by ***five-place*** *logarithms:*

1. 5,359,500.	**5.** .60296.	**9.** .28264.
3. 12.769.	**7.** .00019119.	**11.** .45276.

13. .000051759. **19.** .0000018481. **25.** .85782.
15. .82270. **21.** .93896. **27.** 2.0859.
17. .59205. **23.** 1.7090. **29.** 1657.4.
31. $-$ .0074626. **39.** $-$ .0031429.
33. 10.691. **41.** 37.631.
35. (*a*) 5355.0; (*b*) 3.4715. **43.** 1.2473.
37. (*a*) .72368; (*b*) .94848. **45.** .093068.

Exercise 109. Page 260

Answers, when computed by ***four-place*** *logarithms:*

1. .9870; $-$ .6079. **5.** 324.4. **9.** 397.1.
3. .4131. **7.** 24,290. **11.** 88,450.
13. 7.063 inches.

Answers, when computed by ***five-place*** *logarithms:*

1. .98701; $-$ .60787. **5.** 324.29. **9.** 397.18.
3. .41308. **7.** 24,295. **11.** 88,446.
13. 7.0622 inches.

Exercise 110. Page 262

Answers, when computed by ***four-place*** *logarithms:*

1. 1.464. **3.** 3.595. **5.** 37.79. **7.** 5.608.
9. $x = 2.765$; $y = -$ .2049.

Answers, when computed by ***five-place*** *logarithms:*

1. 1.4639. **3.** 3.5971. **5.** 37.804. **7.** 5.6125.
9. $x = 2.7653$; $y = -$ .20493.

Exercise 111. Page 263

3. $\log_2 10 = 3.3$: $\log_2 5 = 2.3$: $\log_2 .5 = -1$.

Exercise 112. Page 263

Answers, when computed by ***four-place*** *logarithms:*

1. $\log_e 75 = 4.317$. $\log_e 10 = 2.303$. $\log_8 78 = 2.432$.
$\log_{15} 33 = 1.291$. $\log_{12} 1000 = 2.781$.
3. 1.465. **5.** .4343. **7.** 2.303. **9.** $-$ 3.322.

Answers, when computed by five-place logarithms:

1. $\log_e 75 = 4.3176$. $\log_e 10 = 2.3026$. $\log_6 78 = 2.4316$.
$\log_{15} 33 = 1.2911$. $\log_{12} 1000 = 2.7799$.

3. 1.4649. **5.** .43429. **7.** 2.3026. **9.** -3.3219.

Exercise 113. Page 266

1. 360. **3.** (*a*) 120; (*b*) 120. **5.** 240.

7. (*a*) 24; (*b*) *xyzw*; *xywz*; *xzyw*; *xzwy*; *xwyz*; *xwzy*.
yxzw; *yxwz*; *yzxw*; *yzwx*; *ywxz*; *ywzx*.
zyxw; *zywx*; *zxyw*; *zxwy*; *zwyx*; *zwxy*.
wyzx; *wyxz*; *wzyx*; *wzxy*; *wxyz*; *wxzy*.

9. 36. **13.** 36. **17.** 10,080. **21.** 325.

11. 32. **15.** 720. **19.** 600. **23.** 21,772,800.

25. 50,400.

Exercise 114. Page 270

1. 11,880. **3.** 560. **5.** 24. **7.** 720. **9.** 210. **11.** 210.

Exercise 115. Page 273

1. 5. **3.** 462.

5. $\dfrac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43 \cdot 42 \cdot 41 \cdot 40}{13!}$.

7. (*a*) 15; (*b*) 57. **11.** 5040.

9. (*a*) 66; (*b*) 11. **13.** (*a*) 15; (*b*) 330; (*c*) 150.

15. 280. **21.** 216. **27.** 630,630.

17. 10,080. **23.** (*a*) 40; (*b*) 45. **29.** 480.

19. 2880. **25.** 4,233,600. **31.** 288.

Exercise 116. Page 278

1. (*a*) $\frac{20}{31}$; (*b*) $\frac{26}{31}$. **3.** $\frac{1}{2}$. **5.** \$60.

7. \$31.25. **11.** $\frac{8902328}{9628285}$.

9. (*a*) $\frac{2}{5}$; (*b*) \$2. **13.** (*a*) $\frac{6993}{63364}$; (*b*) $\frac{15463}{63364}$.

15. (*a*) $\frac{2}{5}$; (*b*) $\frac{2}{3}$; (*c*) $\frac{1}{7}$; (*d*) $\frac{20}{1001}$; (*e*) $\frac{31}{105}$.

17. (*a*) $\frac{21}{1265}$; (*b*) $\frac{189}{506}$. **19.** (*a*) $\frac{1}{32}$; (*b*) $\frac{1}{32}$; (*c*) $\frac{5}{16}$.

21. $\frac{19}{39}$. **25.** (*a*) $\frac{3}{8}$; (*b*) $\frac{7}{8}$. **29.** (*a*) $\frac{9}{169}$; (*b*) $\frac{1}{26}$.

23. (*a*) $\frac{1}{6}$; (*b*) $\frac{1}{12}$; (*c*) $\frac{1}{6}$. **27.** (*a*) $\frac{2}{15}$; (*b*) $\frac{3}{5}$. **31.** \$$4\frac{8}{9}$.

Exercise 117. Page 284

1. $\frac{1}{15}$. **5.** $\frac{7}{20}$. **9.** $\frac{1}{2}$. **13.** $\frac{1}{8}$.

3. $\frac{3}{20}$. **7.** $\frac{7}{15}$. **11.** $\frac{383}{460}$. **15.** $\frac{2}{5}$.

17. (*a*) $\frac{1}{2}$; (*b*) $\frac{11}{12}$.

19. A's is $\frac{1}{8}$; B's is $\frac{1}{8}$; C's is $\frac{1}{4}$; D's is $\frac{1}{2}$.

21. $\frac{99}{272}$. **23.** $\frac{1}{3}$. **25.** (*a*) $\frac{45}{512}$; (*b*) $\frac{53}{512}$.

27. (*a*) $\frac{3}{32}$; (*b*) $\frac{7}{64}$. **29.** (*a*) $\frac{80}{243}$; (*b*) $\frac{112}{243}$.

31. $\frac{3.(97)^2(7203)}{(7300)^3}$, or .00052265 (computed by use of logarithms).

33. $\frac{69}{1944}$.

Exercise 119. Page 299

1. $c_1\begin{vmatrix} m_2 & r_2 & v_2 \\ m_3 & r_3 & v_3 \\ m_4 & r_4 & v_4 \end{vmatrix} - c_2\begin{vmatrix} m_1 & r_1 & v_1 \\ m_3 & r_3 & v_3 \\ m_4 & r_4 & v_4 \end{vmatrix} + c_3\begin{vmatrix} m_1 & r_1 & v_1 \\ m_2 & r_2 & v_2 \\ m_4 & r_4 & v_4 \end{vmatrix} - c_4\begin{vmatrix} m_1 & r_1 & v_1 \\ m_2 & r_2 & v_2 \\ m_3 & r_3 & v_3 \end{vmatrix}$; etc.

3. -7. **5.** Zero. **7.** -105. **9.** -344.

11. $(x-1)(y-1)(y-x)(xy)$. **13.** $x=3$; $x=2$.

15. $(x-y)(y-w)(w-x)$.

Exercise 120. Page 306

1. $x=1$; $y=-2$; $z=3$; $w=2$.

3. $x=\frac{3}{2}$; $y=1$; $z=-1$; $w=2$.

5. $x=1$; $y=-1$; $z=2$; $w=0$; $v=-2$.

11. $k=1$, or $k=-\frac{14}{5}$.

Exercise 121. Page 313

1. $\dfrac{4}{x-4}-\dfrac{3}{2x+1}$.

3. $\dfrac{1}{x-7}+\dfrac{1}{x+2}$.

5. $\dfrac{2}{y}+\dfrac{2}{y+3}-\dfrac{4}{(y+3)^2}$.

7. $\dfrac{2}{(x-1)^3}+\dfrac{2}{(x-1)^2}+\dfrac{1}{x-1}$.

9. $\dfrac{4}{2x-5}+\dfrac{3-x}{x^2+2}$.

11. $\dfrac{3x+1}{x^2+3}-\dfrac{1}{x+1}+\dfrac{2}{(x+1)^2}$.

13. $-\dfrac{1}{20x+10}-\dfrac{4}{20x-15}-\dfrac{5}{8x^2+8x+2}$.

15. $\frac{1}{x^3} - \frac{2}{x^2} + \frac{1}{x} + \frac{x+5}{x^2+1}$.

17. $\frac{9x-8}{3(x^2+x+4)} - \frac{6x+5}{3(2x^2+2x+5)}$.

19. $\frac{5}{4(x+1)} - \frac{11}{7(x+2)} + \frac{55}{28(3x-1)}$.

21. $-\frac{1}{3(2x+1)} + \frac{2x+1}{3(4x^2-2x+1)}$.

23. $\frac{x+1}{x^2+2} - \frac{2x}{(x^2+2)^2}$.

25. $\frac{-1}{49(x+2)} + \frac{51x-3}{49(2x^2+x+1)} + \frac{4-5x}{7(2x^2+x+1)^2}$.

27. $\frac{19x}{x^2+2} - \frac{5x}{(x^2+2)^2} - \frac{18x}{x^2+3} - \frac{20x}{(x^2+3)^2}$.

Exercise 122. Page 316

1. Amount = \$7997.4. Interest = \$4997.4.
3. Amount = \$6392.4 (by logarithms). Interest = \$2392.4.
5. \$3851.16.
7. \$3331.56.
9. The payment promised in (*a*) is worth \$92.79 more than (*b*), in terms of equivalent sums of money due immediately.

Exercise 123. Page 318

1. .0506.
3. .067.
5. (*a*) .059; (*b*) .088.
7. (*a*) $k = 33.7$; (*b*) $r = .019$.
9. 3.74 years.
11. $r = .0927$.

Exercise 125. Page 322

3. Present value is \$7047; amount is \$23,864.
5. \$15,385.
7. \$1234.
9. Present value: $100\left(\frac{1-(1.02)^{-40}}{(1.02)^2-1}\right)$.

   Amount: $100\left(\frac{(1.02)^{40}-1}{(1.02)^2-1}\right)$.

Exercise 126. Page 324

1. (*a*) \$922.99; (*c*) \$3430.80.
3. \$894.58.
7. 5.8%.
5. Seven annual payments of \$1200 and an additional smaller payment at the end of eight years.

Exercise 127. Page 326

1. \$24,297. **3.** \$210.19. **5.** \$2644.3. **7.** \$2521.7.

9. Use of the amortization method would involve a saving of \$139.14, semi-annually, during the 10 years.

Exercise 128. Page 328

1. (*a*) \$11,666.31; (*b*) \$10,000; (*c*) \$8659.38.

3. \$417.64. **5.** \$1033.96. **7.** 4.5%.

Exercise 129. Page 334

1. 1.72. **3.** 1.1752. **5.** .949. **7.** 1.016. **9.** 5.477. **11.** 4.932.

Exercise 130. Page 339

13. Diverges. **15.** Converges.

Exercise 131. Page 343

1. Converges. **5.** The test fails. **9.** Converges.

3. Converges. **7.** Diverges. **11.** Converges.

Exercise 132. Page 346

1. Converges. **3.** Converges. **5.** Converges.

7. Diverges. **9.** Diverges.

Exercise 133. Page 347

1. Converges if $|x| < 2$, that is, if $-2 < x < 2$.
Diverges for all other values of x.

3. Converges if $|x| < 1$; diverges for all other values of x.

5. Converges if $|x| < .1$; diverges for all other values of x.

7. Converges for all values of x.

9. Converges if $|x| < 1$; diverges for all other values of x.

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